Discussion Paper

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No. 27 9 December 1987

DEPLETION OF LARGE GAS FIELDS WITH THIN OIL LAYERS AND UNCERTAIN STOCKS

BY

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ABSTRACT

The optimal depletion policy for a combined oil and gas field is studied under the assumption that if oil is to be extracted, it has to be depleted before production of gas can start. In particular, focus is put on how uncertainty affects the decision whether or not to extract the oil, and to study the effect of "learning by doing" on this decision. In opposition to conclusions in other literature it is shown that uncertainty alone may change the attitude to oil production. The effect of learning is a more attractive oil production if experience from oil production reduces the uncertainty about gas reserves.

CONTENT

1. INTRODUCTION

A large part of the planning process in oil companies consists of developing optimal depletion policies for oil and gas fields. Central authorities are normally concerned with establishing a tax system which efficiently captures the resource rent, and with the control of the level of activity in the petroleum sector. However, in small countries like Norway, where the petroleum sector amounts to a large part of the total economic activity, the government should also show some concern about the extraction policies of single fields, since the planned extraction path may differ from the optimal one. One reason is that private companies within the petroleum sector often seem to take a shorter view than the social time preference suggests. This is probably why the government allowed for a delay of tax-payments in the Ekofisk water-injection project. Another reason is that since oil and gas extraction is a considerable source of tax income, a non-optimal extraction policy may cause huge losses for the state. Central authorities should therefore assure themselves that all relevant information is taken into account when field development plans are made. If not, they may wish to influence decisions made by private companies.

In this study, we examine how the effect of uncertainty and learning affects the decision whether or not to produce oil from large gas fields with thin oil layers. Such fields constitue an increasing share of total petroleum reserves in Norway. Several of the largest discoveries in recent years are within this class: the recoverable resources in the giant Troll field are approximately 1.3 billion tons of oil equivalents of which 95 per cent is gas, and contribute about 30 per cent of known, remaining Norwegian resources south of the 62nd parallel. Also large discoveries in the northern areas like Snøhvit and Midgard are mainly gas fields where production of the small oil quantities easily might be non-profitable.

If oil is to be produced from these fields, one has to start with extraction of oil, and produce the gas afterwards. Since, in addition, the extraction of the resources are subject to a decreasing uncertainty with respect to remaining reserves, one may ask whether an option value can be attached to them. The theory of option value concerns irreversible decisions with an uncertain outcome, and states that if future learning about the outcome is taken into account, the decisions will be affected, probably in a conservative direction. (See Arrow and Fisher (1974) and Henry (1974)). Thus, the "option value approach" might suggest that oil production can be more profitable than one immediately should think.

However, to apply this approach to the extraction of combined oil and gas fields is probably misleading. This will be discussed in the next section. One may ask what consequences uncertainty and learning have on the extraction path of combined fields compared with a deterministic approach. In particular it is of interest to see under which conditions uncertainty and learning effects might change the decision whether or nor to produce the oil. A deterministic extraction policy for a single field is the basis for later discussion, and a relatively large part of the paper is devoted to the deterministic case.

Most of the results in this paper are known from earlier works. Hoel (1978) and Alfsen (1987) have made analyses of similar topics, but they do not consider the optimal extraction path of oil and gas. In this paper we utilize a dynamic programming approach with continous time and determine an optimal extraction path. This enables us to go through the extraction policy in more detail, and thereby obtain more detailed results.

Though a strict use of the concept of "option value" probably is misleading when applied to combined oil and gas fields, it turns out that the effect of uncertainty and learning is quite similar to the option value case. With reasonable assumptions the extraction of oil is more favorable if learning is taken into account than if it is disregarded. The opposite, however, might be the case if learning about oil conditions is more or less irrelevant for the production of gas. It is also shown, in contrast to previously published results ((Henry (1974), Hoel (1978) and Alfsen (1987)), that the possibilty for profitable oil production is affected by uncertainty alone.

Finally, we will emphasize that only one kind of uncertainty is regarded in this paper - the uncertainty associated with the knowledge of physical conditions of the reservoir, or the amount of recoverable reserves. Other kinds of uncertainty are also of considerable importance in the oil extraction industry, particularily the future price of petroleum. However, learning about the future petroleum price is probably associated with the shape of its probability distribution, and not with a reduction in uncertainty, which we will assume for the estimate of recoverable reserves. There is also uncertainty about future costs and technology, but it is limited since decisions concerning choice of capital equipment to a large extent is taken before production starts. On the Norwegian continental shelf about 2/3 of the present value of costs are development costs, which primarily accrue before production start-up (Lorentsen et al. (1985)).

2. IS THERE AN OPTION VALUE IN COMBINED OIL AND GAS FIELDS?

One may roughly define option value as the value of the advantage obtained by following a flexible development strategy in order to keep future options open. Already Krutilla (1967) discussed the effects of flexibility of future preferences, but a strictly formal introduction to the option value concept was first made by Arrow and Fisher (1974) and Henry (1974). Their main conclusion was that if investment decisions involve irreversible damage to nature, and the future state of the world is uncertain, a more conservative development strategy should be followed if learning about future state is taken into account. Later M. Freeman (1984) and J.R. Miller and F. Lad (1984) has argued that the effect of learning also may be a less conservative extraction policy if there are different ways to gain information, or if different strategies (flexible and fixed) are connected with different costs.

In any case, option value is a consequence of irreversibility, uncertainty and learning. When developing a large gas field with a thin oil layer one must decide whether to produce oil or not before the development is initiated. Oil is produced by expansion of gas in the reservoir, and must therefore be produced before the gas. However, if the gas are of main importance to the profitability, the oil might be too expensive to produce. Changes in the valuation of the oil reserves might change the decision on whether to extract the oil or not. The background for such changes might be different requirements to the return on capital in private companies and in social planning, or it could be the consideration of the option value.

To support the argument that there is an option value connected to the extraction of combined oil and gas fields, it is claimed that future production, or recoverable reserves, are uncertain at production start-up. This uncertainty is reduced during the production period as a consequence of increasing information about the reservoir. Thus, both the uncertainty and the learning conditions are present. Finally, the decision whether to produce oil or not is "irreversible" since oil cannot be extracted unless this is done from the beginning of the production period. If gas is chosen from the beginning, the forces that bring the oil to the surface disappears. This seems to be the background for the argument that there is an option value which should be considered.

However, the irreversibility mentioned here is not the same as the irreversibility condition connected to option values, because the latter involved a closure of future options. When developing combined oil and gas reservoirs there are no future options. The only option to be made is taken at production start up. Later, or in the "next period", there is only one possibility; production of gas, irrespective of what was produced in first period[1]. Thus, the concept of option value, as defined in the above mentioned literature, does not apply to combined oil and gas fields.

None the less, it is shown in earlier works by Hoel (1978) and Alfsen (1987), that the effect of uncertainty and learning will affect the extraction policy. In the following sections the optimal extraction of a resource deposit is discussed. First, the characteristics of the extraction path of a single deposit with certain reserves is discussed. A general cost function is described, and the extraction of two successively produced resources is analysed. Furthermore, the effect of changes in decision parameters is studied. Finally, uncertainty is introduced by a stochastic evolution of remaining reserves. In several earlier works, the problem of "eating a cake of unknown size" is studied (for example Kemp (1976), Gilbert (1978)). Gilbert also take a dynamic programming approach, but he limits himself to study a binary distribution of stocks: Either it is "small" or it is "large". Thus, we cannot study the effect of learning within his framework. We shall make use of a continous distribution in which learning is included, and will also point out how uncertainty and learning affects the relative value of oil compared to gas. Chow (1979) has demonstrated the uncertainty aspect in the case of a Poisson-distribution of stocks. Much of the present analyses is based on his paper.

3. EXTRACTION OF ONE DEPOSIT WITH CERTAIN RESERVES

We consider a company which is to deplete an oil field with certain reserves (R) within a given time interval (O,T). The company is faced with a constant oil price, which is normalized to 1 for simplicity, and a given, convex cost function, $c(u_t)$. u_t denotes extraction at time t. The net income to the company at t can be written as

$$(3.1) \quad f(u_t) = u_t - c(u_t) .$$

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1) One might say that the decision whether to produce at all or not is an option. This does not alter the above argument since one cannot claim the irreversible argument at the same time.

We assume that $f(u_t)$ is bounded, and from the convexity of c, it follows that the income function is concave, i.e. $f''_{uu} < 0$.

The aim is to maximize net present value:

$$(3.2) \quad \text{i)} \quad \max_u \int_o^T f(u_t)e^{-rt}dt$$

s.t.

$$\text{ii)} \quad \dot{x}_t = -u_t \qquad (u_t \geq 0)$$

$$\text{iii)} \quad x_o = R \,,$$

iv) T is given ,

$$\text{v)} \quad x_T = 0 \,,$$

where x_t is remaining reserves at time t. The problem is easily solved by optimal control theory. However, since our aim is to extend the analyses to an explicit treatment of uncertainty and the effect of learning, it is convenient to use a dynamic programming approach (see for instance Dreyfus (1965)).

(3.2) gives an optimal u at t, when x is remaining, and the system can therefore be expressed as a function of x and t alone. Define the function S(x,t), expressing the maximum present value of future income at t, i.e.

$$S(x,t) = \max_u \int_t^T f(u)e^{-r\tau}d\tau \,.$$

Since S(x,t) represents the maximum value, its definition requires that for a small time increment Δt

$$(3.3) \quad S(x_t,t) - S(x_{t+\Delta t},t+\Delta t) \geq \{f(u)e^{-rt}\}\Delta t+o(\Delta t) \,,$$

where $o(\Delta t)$ is a small term which disappears when Δt approaches zero. If u is chosen in order to satisfy (3.2), equality is obtained in (3.3). The Bellman-Dreyfus[1] equation may be developed directly from (3.3), and is

$$(3.4) \quad 0 = \max_u \{f(u)e^{-rt} - uS'_x + S'_t\}$$

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1) This equation is also referred to as Jacobi-Hamilton-Bellman-equation and the fundamental partial differential equation.

Maximizing (3.4) w.r.t. u yields

$$(3.5) \quad f'_u \, e^{-rt} \;=\; S'_x \, .$$

We solve for u, replace the solution in (3.4), and obtain

$$(3.6) \quad f(u^*(x,t))e^{-rt} - u^*(x,t)\cdot f'_{u^*}e^{-rt} \;=\; -S'_t \, ,$$

which is the differential equation that solves the optimal policy path - or extraction path u*.

(3.5) gives the relationship between u and x at all points of time. If we could develop the path of S'_x as time passes, we would also know the optimal policy path via (3.5). An expression for the path of S'_x may be found by differentiating it with respect to time (denoted $\frac{dS'_x}{dt}\Big|_{u^*}$). By the chain rule of differentiation we find

$$(3.7) \quad \frac{dS'_x}{dt}\Big|_{u^*} \;=\; S''_{xt} - S''_{xx}\, u^*$$

since $\frac{dx}{dt} = -u^*$. Differentiating (3.6) with respect to x, and taking (3.5) into account we can show that the right hand side of (3.7) is zero. Thus, S'_x is constant with respect to time. Applying (3.5) we obtain

$$(3.8) \quad f'_{u^*} \;=\; C_0 e^{rt} \, ,$$

where C_0 is a constant depending on x and T. Differentiating f'_u and C_0e^{rt} w.r.t. time yields

$$f''_{uu} \cdot \frac{\partial u^*}{\partial t} \;=\; rC_0e^{rt} > 0 \, .$$

Due to the concavity of f, we must have

$$(3.9) \quad \frac{\partial u^*}{\partial t} < 0 \, .$$

Figure 3.1 and 3.2 shows the extraction path for u with different values of r and x_0.

In figure 3.1 the relations between f'_u, u and t are drawn. In the (f'_{u^*}, u^*) - plane f'_{u^*} is a positive strictly decreasing function of u*,

Figure 3.1 Extraction plane for a single resource.

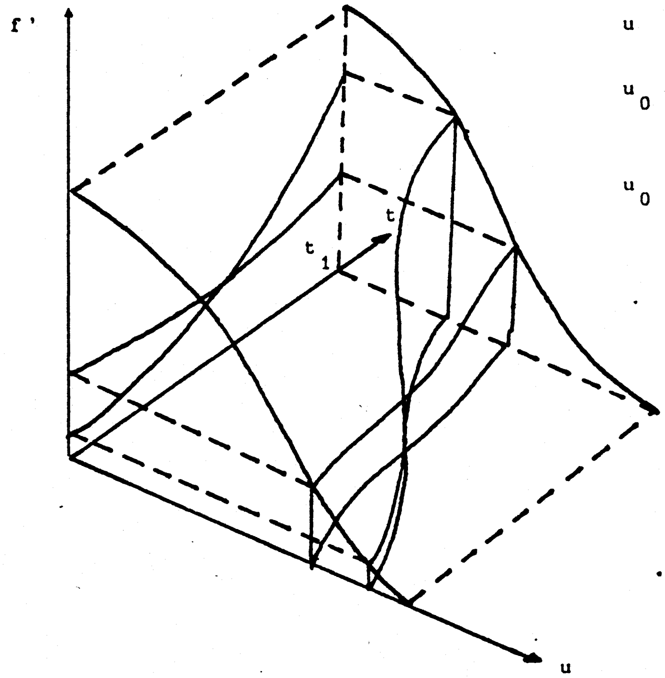
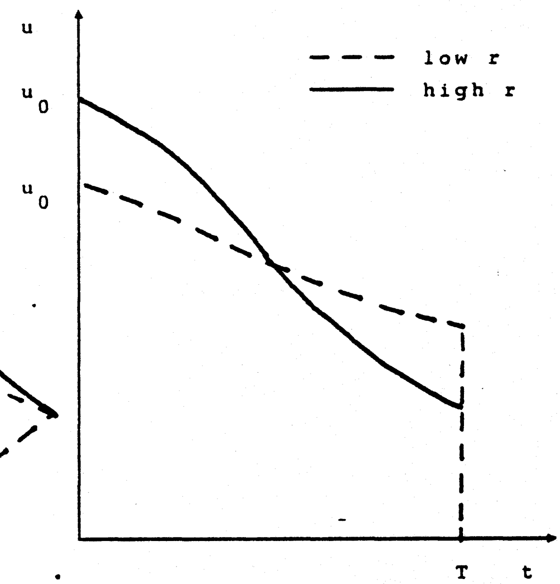


Figure 3.2 Extraction paths at alternative r.



while in the (f'_{u^*},t)-plane f'_{u^*} is monotonically increasing with t, according to (3.8). The extraction path is found in the intersection of the two planes. The figures show the optimal policy for alternative values of r. A high r requires a higher initial extraction (u(o)) and a more rapid decline rate compared to a low r. As expected, a higher initial reserve x_0 requires a higher extraction of u at all t, since f'_u is a strictly decreasing function. With a given reserve, the area below the u(t)-curve drawn in figure 3.2, must be equal (irrespective of required rate of return).

When we later come to the extraction two deposits, it is convenient to have the relationship between the net present value $S_0 = S(x_0,0)$ and the parameters r, x_0 and T in mind. Since increased x_0 implies a higher extraction, u_t, for all t, also income increases with increased x_0, and thus $\frac{\partial S_0}{\partial x_0} > 0$.

The effect on the net present value of increased r follows from optimal setting of the problem: Let u_t^* denote the optimal extraction path at r=r* and u_t^{**} the optimal path at r=r**>r*. Then,

$$\int f(u_t^*)e^{-r^*t}dt \geq \int f(u_t^{**})e^{-r^*t}dt > \int f(u_t^{**})e^{-r^{**}t}dt$$

The first inequality follows from the optimality of u_t^*, which represents the maximum present value among all u-paths, and the second from the fact that for a given u-path the value of the integral decreases with increased r. Thus, we have $\frac{\partial S_0}{\partial r} < 0$.

To see how a change in T affects the net present value we shall consider the case where also T is to be maximized, which seems more realistic than to keep T fixed as long as the company really seeks the maximum yield of its reserves. The optimal choice of T, say T* is found by the equation:

$$(3.10) \quad [f(u_{T*}) - u_{T*}f_u']e^{-rT^*} = 0 .$$

Due to the concavity of f(u), the expression in brackets are positive for all $u_{T*}>0$. Thus, if f(u)=0 for u=0, the optimal terminal point $T^* \to \infty$. From (3.8) it follows that this solution is unique.

However, it is unlikely that f(0)=0. In fact, the cost function and thereby f(u) is probably quite complex, consisting of both investments and operating expenditure. Even if some part of the capital equipment might be transferred to other fields, a large part of the investments must be regarded as "sunk costs". Total costs may therefore be divided into different categories (see Roland (1985)). Here we shall simplify to as great extent as possible, and assume that costs consist of a given amount $\varkappa$ at every point in time in addition to a variable part. $\varkappa$ may be an expression of minimum costs or basic investments for development of a given gas field.

Note that the introduction of $\varkappa$ does not in principle alter the extraction path (i.e. (3.5) is not altered), but the optimal terminal point is altered and it might be that extraction no longer is profitable. The cost function may be expressed as $c(u) = \gamma(u)+\varkappa$, where γ is the variable part of costs. Now the profit function can be written

$$\tilde{f}(u) = u - (\gamma(u)+\varkappa) .$$

We rewrite (3.10) and obtain the optimal condition for T:

$$(3.11) \quad u_T^* \gamma' - \gamma(u) = \varkappa ,$$

which determines a finite T* (although $T^* \to \infty$ is a possible solution). The left hand side of (3.11) is always positive due to the convexity of γ. Increased $\varkappa$ involves a larger required discrepancy between marginal and average variable costs at T*, which gives a higher optimal u_{T*}. As the optimal extraction path, determined by (3.8), requires a steady increase in f_u', the value of C_0 increases by an increase in $\varkappa$, and implies a more "intense" extraction period. If $\varkappa$ is "very large", the optimal u may exceed initial reserves, indicating that extraction is unprofitable. Note also that if the firm requires a "higher" rate of return, r, (3.11) is not affected directly, i.e. the same terminal u_{T^*} is required. Since increased r implies a more intensive extraction, r affects T* negatively.

The optimal terminal T* determines how the net present value S_0 is affected by a change in T in the case of fixed T:

$$\frac{dS_0}{dT} \gtrless 0 \quad \text{dependent on} \quad T \lessgtr T^* . \tag{3.12}$$

4. DEPLETION OF COMBINED OIL AND GAS FIELDS

We now turn to the kind of fields which is the subject of this study - large gas fields with thin oil layers. As mentioned in the introduction, we assume that the oil must be extracted before the gas if it is to be produced at all. We shall take a quite rigorous view and assume that one cannot produce any gas when producing oil. Denoting production of oil in year t as u_t and production of gas in year t as v_t, we state the problem as:

$$\text{(i)} \quad \underset{t_1}{\text{Max}} \left\{ \int_{t_0}^{t_1} f(u_t)e^{-rt}dt + \int_{t_1}^{t_2} g(v_t)e^{-rt}dt \right\} ,$$

$$\text{(ii)} \quad \dot{x}_t = -u_t ,$$

$$\text{(4.1)} \quad \text{(iii)} \quad \int_{t_0}^{t_1} u_t dt = x_0 ,$$

$$\text{(iv)} \quad \dot{y}_t = -v_t ,$$

$$\text{(v)} \quad \int_{t_1}^{t_2} v_t dt = y_0 .$$

g(v) is the profit function for gas, analogous to f(u), the profit function for oil, and we assume the same properties for both. y_t denotes the remaining gas reserves. t_1 denotes the time at which production is switched from oil to gas, and is to be optimized. The optimal paths of u_t and v_t were discussed in the previous section. We concluded that a change in t_1 affected the level of oil extraction, but not the rate of change in extraction over time ((3.8) was still valid), and that a reduction in t_1 if $t_1 \leq T^*$, would reduce the value of the oil.

Clearly, the extraction path of gas is independent of the time for its start up, t_1. Thus, when production of oil is brought to an end, the optimal extraction policy for gas is described in section 3, with terminal time finite or infinite dependent on whether costs are zero at zero production or not, and we denote its present value at t_1 as $\Phi(y_O,r)$. The company would just have to maximize the net present value of the oil with given t_1 if the "switch point" t_1 were known in advance, also described in section 3. The optimal t_1 is determined by maximization of (4.1). We may write the first integral as $\Psi(x_O, r, t_1)$ that is, the net present value of the oil at t is known when initial reserves of oil, the discount rate and t_1 is known. From the earlier discussion, we remember that $\Psi'_{x_O} > 0$, $\Psi'_r < 0$ and $\Psi'_{t_1} \gtrless 0$ dependent on $t_1 \lessgtr T^*$.

The problem (4.1) are now reduced to

$$(4.2) \qquad \max_{t_1} \{\Psi(x_O, r, t_1) + e^{-rt_1}\Phi(Y_O, r)\}$$

which gives the 1st and 2nd order conditions

$$(4.3) \quad \begin{aligned} &\text{(i)} \quad \frac{\partial\Psi}{\partial t_1} = r\, e^{-rt_1}\Phi(y_O, r), \\ &\text{(ii)} \quad \frac{\partial^2\Psi}{\partial t_1^{\,2}} < -r^2 e^{-rt_1}\Phi(y_O, r) \end{aligned}$$

respectively.

The optimized t_1 may now be considered as functions of r, x_O and y_O. (4.3.i) requires a positive $\frac{\partial\Psi}{\partial t_1}$ (if r>0) which implies $t_1 < T^*$ according to (3.10). In words, the production of oil shall come to a halt when the increased net present value from oil caused by increment of planned depletion date equals the loss of interest caused by postponement of gas-income, i.e. marginal income equals marginal loss with respect to the optimal "switch time" t_1.

Increased reserves of gas, enhanced prices or a more favorable cost function for gas may cause an increase in the profitability of the gas (i.e. the net present value of gas at t_1). According to (4.3) the required $\partial\Psi/\partial t_1$ must increase as a result, and since

$$\frac{\partial^2 \Psi}{\partial t_1^2} < 0 ,$$

the optimal t_1 becomes smaller, reducing the value of the oil. As pointed out in section 3, a "very small" terminal depletion date might result in an initial extraction higher than the oil reserves, indicating that extraction of oil is unprofitable.

It is also worth mentioning that the basic investments, called $\varkappa$ in the previous section, probably will be affected by the size of both oil and gas reserves. This makes the link between the profitability of oil and the properties of the gas even closer. "Large" gas reserves with low unit costs makes it expensive to postpone production, and in addition it probably requires a high $\varkappa$, which also affects the cost function of the oil.

It is often claimed that oil companies require a higher rate of return, r, than the social discount rate. It is therefore of some interest to investigate the effects on t_1 of an increment in r. This is a bit more complex than the changes discussed above. (4.3.i) is valid for all r, and we can differetiate with respect to r on both sides. Regarding t_1 as a function of r, x_0 and y_0, we obtain

$$\{\Psi''_{t_1 r} + \Psi''_{t_1 t_1} \frac{\partial t_1}{\partial r}\}dr = \{\Phi(y_0,r)e^{-rt_1} + r\Phi'_r e^{-rt_1} - rt_1 e^{-rt_1}\Phi(y_0,r)$$

$$- r^2 \frac{\partial t_1}{\partial r} e^{-rt_1}\Phi(y_0,r)\}dr$$

Rearranging terms yields

$$(4.4) \qquad \frac{\partial t_1}{\partial r} = \frac{\{(1-t_1 r)\cdot\Phi(y_0,r) + r\Phi'_r\}e^{-rt_1} - \Psi''_{t_1 r}}{\Psi''_{t_1 t_1} + r^2\Phi(y_0,r)e^{-rt_1}}$$

From (4.3 ii) we know that the denominator is negative. Let us also assume that $\Psi''_{t_1 r}$ is "small" (i.e. close to zero). If so, the first term in the brackets, { }, will be positive if $rt_1 < 1$. With a low t_1 and r, the nominator in (4.4) will be positive and $\partial t_1/\partial r < 0$, i.e. if the profit ability of gas is "high enough". In the the opposite case, where oil are dominating the sign of $\partial t_1/\partial r$ will be positive.

5. DEPLETION WHEN RESERVES ARE UNCERTAIN

Up to now, reserves at $t=0$ have been assumed certain. That is, the oil company knows what is to be extracted. Unfortunately, plans for extraction of gas/oil-fields are not made under such favorable circumstances - one must expect continous revisions of estimates of reservoir parameters and thereby remaining reserves. Properly speaking, the reserves are not known with certainty before the last cubic feet of gas has been produced. Information on which reserve estimates are based, are continuously updated during the production period. This may lead to positive or negative revaluations, but the uncertainty will probably decrease monotonously.

To implement this uncertainty into the analyses, the expression for the evolution of remaining reserves $x(t)$ is relaxed. In sections 3 and 4 the reduction in x at t was set equal to extraction $u(t)$. Now, we will assume that in addition to oil or gas also information is gained or produced which may lead to a revision of the estimate of remaining reserves. Thus, for small time increments, Δt, the evolution of remaining reserves can be written

$$(5.1) \qquad x(t+\Delta t) - x(t) = -u\Delta t + z(\Delta t+t) - z(t) ,$$

where $z(t+\Delta t) - z(t)$ is a random term, we can call it dz. It is assumed that dz has a known probability distribution, $p(dz; x,u,t)$, with mean

$$E(dz) = \int_{-\infty}^{\infty} dz\, p(dz; x,u,t)\, d(dz) = 0$$

and variance

$$\mathrm{var}(dz) = \int_{-\infty}^{\infty} (dz-E(dz))(dz-E(dz))p(dz; x,u,t)\, d(dz) .$$

Now, let the variance of dz at $\Delta t=1$ be σ^2. For smaller time increments, $\Delta t < 1$, we simply divide the time interval into segments of equal length, say n, so that $\Delta t = \frac{1}{n}$. If we assume that the n successive increments $z(t+\Delta t)-z(t)$, $z(t+2\Delta t)-z(t+\Delta t), \ldots, z(t+1)-z(t+(n-1)\Delta t)$ are statistically independent, even for small of Δt, the variance of their sum is equal to the sum of the variances, implying that

$$(5.2) \qquad \mathrm{var}(z(t+\Delta t) - z(t)) = \frac{1}{n}\sigma^2 = \sigma^2\Delta t$$

Since the variance is propotional to the time increments, the randomness neither "washes out" nor "smoother" the process.

Note that σ is not neccessarily a constant, but might be a function of x, u or t. In the case of resource depletion under uncertainty, σ will at least be dependent on x. If $\sigma = \bar{\sigma}x$, where $\bar{\sigma}$ is a constant, we may interprete the randomness as constant uncertainty with respect to remaining reserves, and we will refer to it as the "standard uncertainty case". However, as mentioned one must expect a certain learning effect as a result of production. One way to include this into the analyses is to assume decreasing uncertainty with respect to reserves. With reduction in remaining reserves, the variance decreases with more than $\bar{\sigma}x$, or $\sigma'_x > \bar{\sigma}$.

There are other ways to include the effect of learning. One alternative is to express it in terms of t (Alfsen (1987)), or both x and t to account for a general progress in the knowledge about the geology in the area. A more satisfying representation of improved information would be to let prior-probability densities represent the probability distributions (see for example Dreyfus (1965)). In this paper we will concentrate on the case where $\sigma = \sigma(x)$. Extending the analyses to the case where $\sigma = \sigma(x,t)$ is a simple matter. To apply prior probabilities would require a more complex analyses than the one used here.

We define the optimal policy function S(x,t), similiar to the one in section 3. According to (5.1) the development of x is written

$$(5.3)\qquad dx = -u\,dt + dz$$

with the properties described above. Analogously to (3.3) we obtain "the recurrence relation":

$$(5.4)\qquad S(x,t) = \max_u E\{f(u)e^{-rt}\Delta t + S(x-u\Delta t+dz+o\Delta t,\ t+\Delta t)\}\ .$$

Using Itô's difference rule for stochastic equations, we obtain after some manipulations

$$(5.5)\qquad 0 = \max_u E\{f(u)e^{-rt} + S'_x(-u+dz) + S'_t + \frac{1}{2} S'_{xx}(-u+dz)^2\}\ .$$

Taking the expectation yields directly the Bellman-Dreyfus equation:

$$(5.6)\qquad 0 = \max_u \{f(u)e^{-rt} - uS'_x + S'_t + \frac{\sigma^2}{2} S''_{xx}\}$$

where σ is a function of x only. (5.6) corresponds to (3.4) in the deterministic case. Maximization of (5.6) with respect to u yields the optimal condition

$$(5.7) \quad f'_u \, e^{-rt} = S'_x \, .$$

(5.7) is similar to the basic equation for the discussion of the optimal policy path when the amount of reserves was certain (equation (3.5)). There, the optimal path of u was developed by differentiation of S'_x with respect to time. Now, since S'_x is a function of the stochastic process (5.3), we need to find the expected rate of change in S'_x.

Kushner (1967) provides a solution to this, if the process dx is a right continous strong Markov-process. Let the function G(x,t) be in the domain of a weak infinitesimal operator $\tilde{A}$. $\tilde{A}G(x,t)$ can be interpreted as the average rate of change in the process $G(\cdot)$ in a given time interval. Dynkin (1965) has shown that

$$(5.8) \quad EG(x_{t+\Delta t}, t+\Delta t) - G(x_t, t) = E \int_t^{t+\Delta t} \tilde{A}G(x,s)\, ds \, .$$

If the Itô process (5.3) has the above mentioned properties, the integral on the right hand side can be found by expansion of G:

$$(5.9) \quad \int \tilde{A}G(x,s)ds = G'_x \, dx + \frac{1}{2} G''_{xx} \, d^2x + G'_t \, .$$

The expected rate of change in S'_x can now be found by the same procedure:

$$(5.10) \quad \int_t^{t+\Delta t} \tilde{A}S'_x(x,\tau)d\tau = S''_{xx}(-u+dz) + \frac{1}{2} S'''_{xxx}(-u+dz)^2 + S''_{xt} \, .$$

We let $t \rightarrow 0$, take expectations and obtain:

$$(5.11) \quad EdS'_x(x,t) = (-uS''_{xx} + \frac{\sigma^2}{2} S'''_{xxx} + S''_{xt})dt \, .$$

Differentiating (5.6) w.r.t. x, and noting that u = u(x), we also get

$$(5.12) \quad 0 = f'_u \frac{du}{dx} e^{-rt} - S'_x \frac{du}{dx} - uS''_{xx} + S''_{xt} + \sigma\sigma'_x S''_{xx} + \frac{\sigma^2}{2} S'''_{xxx} \, .$$

Since, along the optimal path $S'_x = f'_u e^{-rt}$, we are left with:

$$(5.13) \quad S''_{xt} = uS''_{xx} - \sigma\sigma'_x S''_{xx} - \frac{\sigma^2}{2} S'''_{xxx} \, .$$

Replacing S''_{xt} in (5.11) yields:

(5.14) $EdS'_x = (-\sigma\cdot\sigma'_x S''_{xx})dt$.

(5.14) confirms the deterministic solution that if $\sigma=0$, $EdS'_x = 0$. In the stochastic case ES'_x will change as time passes, and the rate of change will depend both on the present uncertainty, expressed by the expected average deviation σ, and the learning effect σ'_x. Let us first examine in what direction EdS'_x moves. Differentiating both sides of (5.7), we find:

(5.15) $S''_{xx} = e^{-rt} f''_{uu} \frac{du}{dx}$.

If x developes without uncertainty, we found in section 3 that $\frac{du}{dx} > 0$. It is intuitively acceptable to assume this to be valid also when the evolution of x is random. Since $f''_{uu} < 0$, we thus have $S''_{xx} < 0$ and

(5.16) $EdS'_x > 0$.

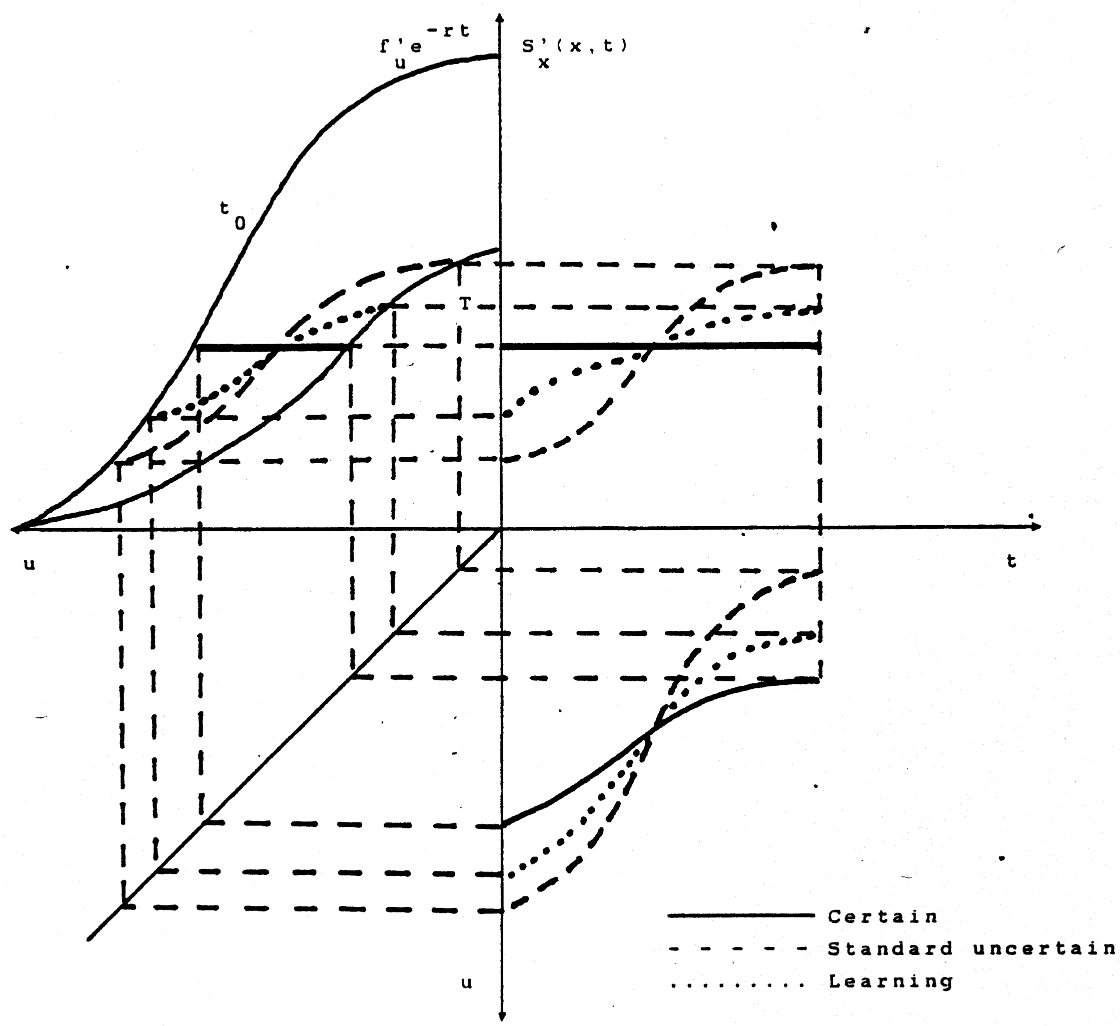
The three cases to be studied are:

The certain, or deterministic case (denoted c), with $\sigma = 0$.
The "standard uncertainty" case (denoted s), with $\sigma = \bar{\sigma}x$.
The learning case (denoted l), where $\sigma = \sigma(x)$ is a convex function.

To compare the two latter cases we need to know the difference between $\sigma(x)^s$ and $\sigma(x)^l$. Since our starting point is how the optimal behaviour is determined by different approaches, we will assume that the uncertainties on the reserve estimates are equal when production starts, i.e. $\sigma(x_0)^s = \sigma(x_0)^l$. We cannot compare the three different cases directly without knowing the explicit solution to the problem. However, we know that (5.16) is valid in the uncertain cases, and that $EdS'^{\,l}_x$ approaches zero as $t \to T$, since $\sigma \to 0$ when $x \to 0$. Since $\sigma(x)^l$ is a convex function in the learning case, $EdS'^{\,l}_x$ will be steeper than $EdS'^{\,s}_x$ in the first years, but sooner or later (probably very soon) the reduction in uncertainty will cause the expected increase in $EdS'^{\,l}_x$ to be lower than the increase in $EdS'^{\,s}_x$.

The extraction paths are illustrated in figure 5.1. The north-west quadrant displays the relation between $f'_u e^{-rt}$ and u with one curve for each t. As time passes, the curve shifts downwards, and in the figure the curves representing t=0 and t=T are drawn. The vertical axis also represent the value of S'_x, according to (5.7), and the north-east quadrant displays its value at different t. The expected evolution of S'_x is found by the corre-

Figure 5.1 Extraction paths for one resource in the certain, the standard uncertain and the learning case.



spondance between u and S'_x in the two upper quadrants. The policy path is found by transferring u* and t from the two upper quadrants to the south-east quadrant. Note that the level of S'_x must be chosen so that the area $\int_0^T u(\tau)\, d\tau$ is equal in all three cases.

It is seen from the figure that the standard uncertainty case results in a "steeper" extraction path with higher initial extraction level and lower level when production terminates, compared to the certain case. When learning is introduced, the optimal path "falls in" between the standard uncertainty and the certain paths. It can be shown that if the learning effect is extremely strong in the beginning, production in the learning case may start at a higher level than in the standard case. In the sequel, such strong learning effects will be disregarded. Note that even though the uncertainty diminishes as $t \to T$ in the learning case u_T^c does not equal u_T^s. The same applies for u_0^c and u_0^s, even though the value of $\sigma(x_0)$ is assumed to be the same in the two cases.

In section 4, the question whether to produce oil or not was reduced to a study of the relative net present value (NPV) of oil versus that of gas. It is therefore neccessary to find how NPV is affected by uncertainty and further what the effect on learning is. We have seen that the difference between the tree cases may be expressed by the steepness of the extraction curves. In the appendix it is shown that for two periods the highest possible NPV among all extraction paths is obtained in the certain case. Further, it is shown that for two periods, NPV decreases the "steeper" the path is, given that it is "steeper" than the certain path. The results may also be extended to n periods. Thus the certain path must give the highest NPV of the three cases.

Comparing the standard uncertainty case with the one in which learning is involved, we remember that just after production start up, the learning profile may be steeper than the standard profile. The above argument is therefore not strictly sufficient for saying that the learning case yields a higher net present value than the standard case. The appendix shows for a two-period case that if the learning path lies closer to the certain path than does the standard uncertain path, the learning case will be the most profitable of the two. Assuming that this is valid for all t, we may conclude that

$$(5.17) \quad NPV^c > E(NPV^l) > E(NPV^s) \; .$$

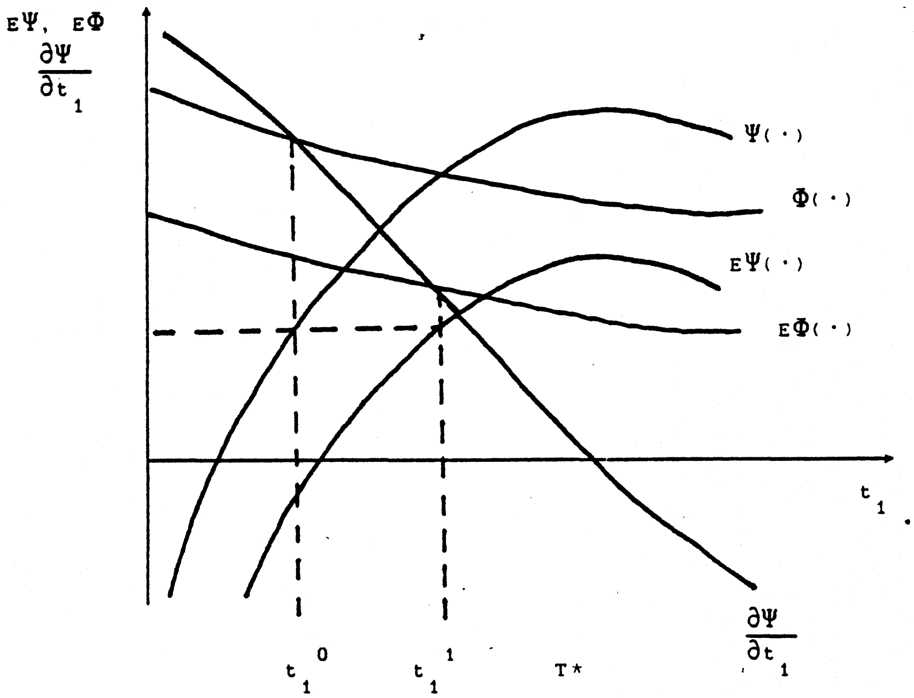
We can call the difference between the net present value in the standard uncertainty and the learning case the value of learning.

The case of two resources, oil (x) and gas (y) may be enlightened by the above results together with the results in section 4, where it was shown that the extraction path of oil was influenced by the relative profitability of oil and gas. In Hoel (1978) and Alfsen (1987) it is stressed that the decision whether to produce oil or not is not affected by uncertainty alone. Is this result valid also in the present analyses?

We use the same notation as in section 4. Thus, $\Psi(x)^c$ is the optimal net present value of oil in the certain case, and $\Phi(y)^c e^{-rt_1}$ the optimal net present value of gas in the certain case. t_1 is the optimal "switch time" from oil to gas production. The standard and the learning case may be denoted likewise, and the first order condition for optimal t_1 is similar in all three cases:

$$(5.18) \quad \frac{\partial \Psi^i}{\partial t_1^i} = re^{-rt_1} E\Phi(y)^i \, , \qquad i = c,s,l \; .$$

Figure 5.2 Switch time from oil to gas in the certain and the uncertain case.



Since

$$\frac{\partial^2 \Psi^i}{\partial t_1^i} < 0 \qquad\qquad i = c,s,l.$$

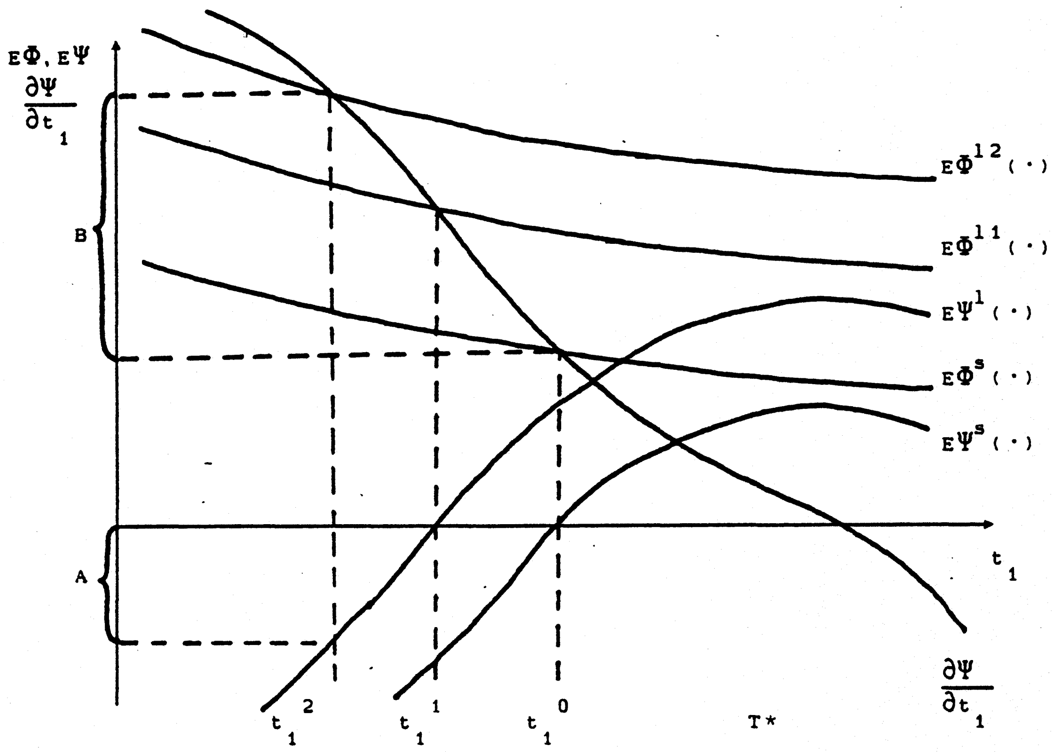
according to the second order condition, a large $E\Phi(y)^i$ implies a small t_1, indicating that oil is unprofitable.

The first question is whether t_1 possibly could be influenced when comparing the certain and the standard uncertainty case. It is seen from (5.18) that it would, unless $\Psi_t'^{c}/\Phi^c = \Psi_t'^{s}/E\Phi^s$, which is most unlikely. We shall assume that $\Psi_t'^{i}$ is affected only to a very limited extent.

To analyse and compare the three cases, we shall first concentrate on the certain and an uncertain case. Since f(u) is concave, the solution of the optimal path of u at <u>given</u> t_1 is unique, which implies that according to the second order condition, $\Psi(\cdot)$ is concave w.r.t. t_1. This is shown in figure 5.2, where the values of $\Psi(\cdot)$ in the certain and an uncertain case is drawn as two concave curves. Since f(u) is bounded and concave, $\Psi(\cdot)$ may become negative if the period $(0,t_1)$ is "too short".

The marginal income of gas is represented by the downward sloping curves in figure 5.2. The difference between the certain $\Psi(\cdot)$ and the uncertain $E\Psi(\cdot)$ case is a lower marginal income of gas in the certain case. This follows fram the fact that the net present value decreases when uncertainty is introduced (most in the standard uncertainty case, and somewhat less in the learning case).

Figure 5.3 The effect of learning compared to a standard uncertain case.



However, the effect on the extraction policy of the whole field is determined also by $\partial\Psi/\partial t_1$, according to (5.18). In the certain case, we found that $\partial\Psi/\partial t_1$ was positive, zero or negative dependent on whether t_1 was less, equal or larger than the optimal terminal time T*. How $\partial\Psi/\partial t_1$ is affected by the three cases also depends on how $\Phi(\cdot)$ is affected, but we will assume that it changes to a very small extent. For the sake of simplicity we will limit ourselves to study the special case where $\partial\Psi/\partial t_1$ is identical in all three cases. A shift from the certain case to the learning and standard uncertainty cases then results in vertical, downward shifts in the $E\Psi(t_1;r,x_0)$-curves. Thus the downward sloping $\partial\Psi/\partial t_1$-curve in the figure represents all certain and uncertain cases.

The optimal t_1 is found by the intersection of the marginal income curves for oil and gas w.r.t. t_1. If the present value of gas at t_1 is $\Phi(\cdot)$ both in the certain and the uncertain case (i.e. only oil reserves are uncertain), the optimal "switch time", t_1^0, becomes equal in both cases. Oil production is profitable in the certain case, but not in the uncertain one. However, had it not been for the very existence of gas, the uncertain case would also give profitable oil production since the optimal terminal time for oil alone is T*. If the net present value of gas also is affected by uncertainty, the value of $E\Phi(\cdot)$ shifts downwards, and the optimal switch time increases to t_1^1, and oil production again turns attractive. Whether oil production is less attractive in the uncertain case than in the certain one depends on to what extent the values of oil and gas are affected by the uncertainty of reserve estimates. For example, if the value of gas is affected more than shown in figure 5.2, the the oil is more attractive in

To see how learning affects the attractivness of oil production we shall compare the learning case with the standard uncertain case like in figure 5.2. The learning case income-curves lie above the standard uncertain case income-curves for both oil and gas. The distance between them depends on how strong the learning effect is. In figure 5.3, t_1^0 denotes the switch time in the uncertain case. The learning case is indicated by two alternatives for $E\Phi^1$. If oil production does not contribute to the knowledge of uncertain parameters determining the estimates of gas reserves, the switch time in the learning case is t_1^1. If oil production actually contributes to such knowledge, the increase in $E\Phi^1$ results in an even shorter period for oil production, t_1^2. The immediate effect of the learning approach is a lower uncertainty on the average, and may be discussed similiarly to the effect of uncertainty in figure 5.2. I 5.3 this "first effect" is represented by $E\Phi^{11}$, and the two cases turn out to be equally attractive for oil production: The net present value is zero in both cases.

The second effect which is to be considered when learning is introduced, is that oil production may reduce the uncertainty of gas at t_1, and thereby increase the value of gas, represented by $E\Phi^{12}$. If this gain exeeds the reduction in profitability of oil, the value of oil is increased in the learning case. To compare the standard and the learning case it is necessary to compare the "loss" of producing oil, A in the figure, with the gain in profitability of gas, B.

Three conclusions can now be drawn: First, all three approaches (certain, standard uncertain or learning) have different optimal solutions, except in special cases. Second, whether the learning case is more favorable to oil production than the standard uncertain case depends upon how much the uncertainty in the estimate of gas reserves is reduced during the oil production period. Third, even if oil production is unprofitable alone, it may be worthwhile to produce it if it enhances the profitability of gas sufficiently. The approach in this paper may also be applied to the problem of optimal exploration efforts, and this second argument is exactly the explanation for exploration expenditure.

Comparing the learning case with the certain one, we note that both the immediate effect (reduced value of $E\Phi$) and the learning effect on gas turns oil production more favorable when learning is taken into account.

When discussing the difference between the three cases, it was assumed that $\partial\Phi/\partial t_1$ is unaffected by uncertainty and learning. The above conclusions are weakened if this curve shifts upwards with increased uncertainty.

6. CONCLUDING REMARKS

Option value is the value of following a flexible development strategy when irreversible decisions are to be made. The option value approach to development strategies does not apply directly to large gas fields with thin oil layers, since there are no options in future periods. However, the effects of taking uncertainty and learning into account when deciding an optimal extraction policy are in many respects similar to those obtained in the option value literature. The dynamic programming approach has, however, the advantage that results from previously published papers are gathered in one model, an in addition a "new" result occur.

The relative value of oil and gas determines whether the oil is to be produced or not. Oil production might be abandoned even if it is profitable, taken isolated, because the cost of a postponement of gas may exceed the income from extraction of oil. When any kind of uncertainty is taken into account, the value of postponing gas will be changed, while the value of changing the production period for oil is probably affected to a limited extent. Thus, in opposition to conclusions in earlier literature, the possibility of profitable oil production is changed by uncertainty alone.

The impacts of learning may be that the production of oil becomes more favorable. Even if the expected net present value of oil is negative, it may be produced, mainly to gain further information about gas-conditions. However, if the production of oil does not contribute much to the infor mation about the gas conditions, the opposite might be the case: The learning approach turns production of oil less favorable.

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APPENDIX

THE IMPACT ON THE NET PRESENT VALUE OF UNCERTAINTY AND LEARNING.

This appendix shows that a two-period path, divergent from the optimal path, gives a lower net present value (NPV) the larger the difference between the two paths are. This is the background for the conclusion drawn in (5.17).

Along the optimal extraction path the condition

(A1) $$C_0 = f_u' e^{-rt}$$

must be fullfilled. Let u_0 and u_1 denote an optimal, two-period path, and b(t) the discount factor. Then (A1) can be written

(A2) $$f_{u_0}' = f_{u_1}' b(1)$$

Denote the difference between an arbitrary path and the optimal one, du (du $\neq$ 0). Since total stock is equal in the two cases we have

(A3) $$du_0 = -du_1$$

NPV for the arbitrary path may be written

(A4) $$NPV(du) = f(u_0+du) + f(u_1-du)b(1)$$

If du > 0, an increase in du may be interpreted as a "steeper" path, so if NPV(du) decreases with increased du, then (5.17) will be fulfilled (in the two-period case). Derivation of NPV(du) gives:

(A5) $$NPV'(du) = f_u'(u_0+du) - f_u'(u_1-du)b(1)$$

By differentiation we obtain

(A6) $$dNPV'(du) \approx f_u'(u_0) + f_{uu}''du - f_u'(u_1)b(1) + f_{uu}''du\ b(1)$$

Because of (A2), the first and third term in (A6) are zero, and we are left with

(A7) $$NPV'(du) = f_{uu}''du + f_{uu}''du\ b(1) < 0$$

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