The dual approach for measuring multidimensional deprivation: Theory and empirical evidence

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ABSTRACT

This paper is concerned with the problem of ranking and quantifying the extent of deprivation in multidimensional distributions of dichotomous deprivation variables. To this end, we introduce a family of measures of deprivation justified on the basis of dual social evaluation functions.

Two alternative criteria of second-degree deprivation count distribution dominance are shown to divide the proposed family of deprivation measures into two separate subfamilies, which can be justified by a combination of correlation increasing and count neutral rearrangements.

Based on EU-SILC data, we show that application of the proposed measures might lead to conclusions that differ from those attained by standard cut-off measures, and that results based on cut-off measures are more sensitive to the choice of specific measure.

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1. Introduction

Multidimensional poverty and inequality is not a new topic in economics, but the extent of the literature has been rather modest until the recent 10–15 years where most papers have considered cases with continuous variables. In this paper, we focus on situations where the multiple attributes in which an individual can be deprived are represented by dichotomized variables. The number of dimensions for which each individual suffers from deprivation may therefore be summarized in a "deprivation count" (see Atkinson, 2003).[1] The purpose of this paper is *not* to discuss the justification for counting the deprivation indicators; we take it for granted by referring to the extensive practice of statistical agencies to publish such data; normally summarized by

three summary measures: The proportion of people suffering from at least one deprivation indicator, the proportion of people suffering from all deprivation indicators and the average number of deprivations in the population. The importance of collecting such data has also been emphasized by the European Union as part of the European 2020 Agenda measures. Therefore, EUROSTAT (the Statistical Agency of the EU) collects counting data on a regular basis, as part of the EU-SILC microdata on level of living. These facts form a motivating background for investigating deprivation count distributions.

Being deprived on a single dimension could result from the combination of a threshold and a continuous or discrete variable (e.g. income below the poverty line or fewer than a specific number of healthy days for a year). In what follows, it is supposed that available data only contain information on whether an individual is deprived or not in each dimension; the variables are dichotomous. This simplification allows us to delve into the question of how to measure (overall) deprivation in a country. As for the analysis of poverty in multidimensional distributions of continuous variables, the order of aggregation is of crucial importance for the measurement of deprivation in count distributions. Data limitations might in some cases only allow to first aggregate across

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[1] Bossert et al. (2013) and Lasso de La Vega and Urrutia (2011) provide alternative axiomatic foundations of deprivation measures for multidimensional distributions of dichotomized variables.

individuals for each dimension and next aggregate the dimension-specific proportions into an overall measure of deprivation (or poverty). The Human Poverty Index (HPI) is a prominent example of this approach.[2] However, when data provide information on all dimensions for the same individuals it is more attractive to employ the opposite order of aggregation. Otherwise, essential information about the association between deprivation indicators would have been lost.[3] First, by aggregating across dimensions for every individual, a "deprivation count" representing the number of dimensions for which the individual suffers from deprivation is identified. Second, by aggregating across individuals, we obtain a count distribution, which will form the informational basis of the methods introduced in this paper.

Atkinson's (2003) illuminating discussion on the relationship between social welfare, measurement of deprivation and association between different attributes has formed the motivation and inspiration for this paper.[4] However, as opposed to the methods discussed by Atkinson (2003), which can be justified by the "primal independence axiom" of the expected utility theory,[5] the methods proposed in this paper rely on an alternative independence axiom called the "dual independence axiom" by Yaari (1986). The dual independence Axiom in combination with some standard axioms is shown to characterize a general family of deprivation measures. These measures are obtained by aggregating a transformation of the count distribution function over the range of counts and are moreover shown to admit a linear decomposition with respect to the mean and dispersion of deprivation counts, where the choice of the dispersion measure depends on the preferences of a social planner. More precisely, the functional form of the dispersion measure (i.e. preference function) reveals whether the concern of the social planner is turned towards those people suffering from deprivation on all dimensions (convex "preference" function) or those suffering from at least one dimension (concave "preference" function). This distinction is also demonstrated to be captured by two alternative partial orders; second-degree upward and downward count distribution dominance, which refine the trivial ranking of deprivation count distributions provided by Pareto dominance (or first-degree stochastic dominance).

A normative justification of the dominance criteria is provided by combining a correlation increasing rearrangement (see e.g. Atkinson and Bourguignon, 1982; Tsui, 1999; Bourguignon and Chakravarty, 2003 and Aaberge and Brandolini, 2015) with an alternative rearrangement called count neutral rearrangement. Count neutral rearrangement is a rearrangement that does not affect the deprivation count of individuals; it solely affects the allocation of deprivations between dimensions. As is demonstrated in this paper, the combination of correlation increasing rearrangement and count neutral rearrangement can also be used to justify the division of the general family of dual deprivation measures into two subfamilies, determined by whether the preference function of the social planner is convex or concave.

The common approach for measuring multidimensional deprivation in the literature is to use cut-off measures defined by the proportion of individuals suffering from z or more dimensions for some cut-off z (e.g. Guio et al., 2017). An essential difference between the cut-off approach used by Guio et al. (2017) and our approach is due to different informational basis. Our methods rely on the entire deprivation count distribution, whereas the cut-off methods ignore information from the left tail of the count distributions. Moreover, we have introduced methods that differ in their sensitivity to changes that take place in the lower, the central and the upper tail of the count distribution. Thus, an interesting question is whether the methods introduced in this paper produce results that differ from those obtained by application of cut-off measures. To compare the dual deprivation measures with cut-off measures, we use alternative cut-off and dual deprivation measures to assess the effects of the Great Recession on material deprivation in 30 European countries. The count data in question are defined by indicators of material deprivation collected by the EU Statistics on Income and Living Conditions (EU-SILC) project, which assess whether an individual is suffering from material deprivation on 10 different dimensions. We show that the dual deprivation measures provide results that differ from the results produced by the cut-off measures in 6–24% of the country-specific comparisons, depending on the specific chosen measure. More importantly, we show that conclusions as to whether material deprivation increased or decreased between two years are robust to the choice of measure in only 29% of the cases when using cut-off measures, while conclusions attained by application of dual deprivation measures are robust in 65% of the cases for concave preference functions and in 40% of the cases for convex preference functions.

The paper is organized as follows. Section 2 presents second-degree upward and downward dominance criteria as suitable refinements of first-order stochastic dominance. These criteria differ by capturing alternative ethical views of a social planner, who either give priority to individuals suffering from few or from many deprivations. Moreover, we introduce a family of deprivation measures on the basis of axioms used for justifying measures of social welfare. The deprivation measures are shown to admit a useful decomposition with respect to the extent and the dispersion of deprivation counts. Section 3 introduces association rearrangement principles, which are shown to justify second-degree upward and downward dominance and two subfamilies of dual deprivation measures as criteria for ranking deprivation count distributions. Section 4 provides an application of the introduced methods to assess the effect of the Great Recession on material deprivation in 30 European countries, comparing the dual deprivation measures with cut-off measures commonly used in the literature. Section 5 provides a summary of the paper and a discussion of possible developments.

2. Ranking distributions of deprivation counts

We consider a situation where individuals might suffer from r different dimensions of deprivation. Let X_i be equal to 1 if an individual suffers from deprivation in dimension i and 0 otherwise. Moreover, let $X = \sum_{i=1}^{r} X_i$ be a random variable with cumulative distribution function F and mean μ, and let F^{-1} denote the left inverse of F, i.e. $F^{-1}(t) = \inf\{k : F(k) \geq t\}$. Thus, $X = 1$ means that the individual suffers from one deprivation, $X = 2$ means that the individual suffers from two deprivations, etc. We call X the deprivation count and F the deprivation count distribution. Furthermore, let $q_k = \Pr(X = k)$ which yields

$$F(k) = \sum_{j=0}^{k} q_j, \quad k = 0, 1, 2 \dots, r \tag{2.1}$$

with mean $\mu = \sum_{k=1}^{r} k q_k$. Note that the mean admits the following alternative expression[6]

$$\mu = r - \sum_{k=0}^{r-1} F(k). \tag{2.2}$$

To anticipate the results of Section 2.2, note that expression (2.2) reveals the basic structure of the dual approach: Replacing F in Eq. (2.2) with a transformation of F, say $\Gamma(F)$, corresponds to replace the mean with an "equally distributed equivalent number of deprivations",

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[2] See Anand and Sen (1997).

[3] The importance of accounting for the association between dimensions in analyses of multidimensional inequality and poverty has been underlined by e.g. Atkinson and Bourguignon (1982), Tsui (1999), Atkinson (2003), Bourguignon and Chakravarty (2003) and Alkire and Foster (2011).

[4] See also Duclos et al. (2006).

[5] The primal approach has been considered by Alkire and Foster (2011) and Aaberge and Brandolini (2014, 2015).

[6] In fact, $\mu = \sum_{k=1}^{r} k q_k = r - \sum_{k=0}^{r}(r-k)q_k = r - [r q_0 + (r-1)q_1 + \cdots + q_{r-1}] = r - \sum_{k=0}^{r-1} \sum_{j=0}^{k} q_j = r - \sum_{k=0}^{r-1} F(k)$.

which will depend on the normative judgements captured by the shape of the "preference" function Γ.

2.1. Partial orders

As for distributions of continuous variables (like income) comparisons of count distributions can be achieved by employment of appropriate dominance criteria. The condition of first-degree dominance, i.e. $F_1(k) \geq F_2(k)$ for all $k = 0, 1, 2, ..., r - 1$ and the inequality holds strictly for some k, justifies the claim that F_1 exhibits less deprivation than F_2.

To deal with situations where deprivation count distributions intersect, weaker dominance criteria than first-degree dominance are called for. As will be demonstrated below, it will be useful to make a distinction between aggregating across count distributions from below and from above.[7] We first introduce the "second-degree downward dominance" criterion.

Definition 2.1A. A deprivation count distribution F_1 is said to second-degree downward dominate a deprivation count distribution F_2 if

$$\sum_{k=s}^{r-1} F_1(k) \geq \sum_{k=s}^{r-1} F_2(k) \quad \text{for } s = 0, 1, ..., r-1$$

and the inequality holds strictly for some s.

A social planner who implements second-degree downward count distribution dominance is especially concerned with those people who suffer from deprivation on many dimensions. However, an alternative ranking criterion that focuses attention on those who suffer from deprivation on few dimensions can be obtained by aggregating the deprivation count distribution from below.

Definition 2.1B. A deprivation count distribution F_1 is said to second-degree upward dominate a deprivation count distribution F_2 if

$$\sum_{k=0}^{s} F_1(k) \geq \sum_{k=0}^{s} F_2(k) \quad \text{for } s = 0, 1, ..., r-1,$$

and the inequality holds strictly for some s.

Note that second-degree downward as well as upward count distribution dominance preserves first-degree dominance since first-degree dominance implies second-degree downward and upward dominance.

The following example illustrates the difference between the two principles: Consider two counting distributions F_1 and F_2. In distribution F_1, individual i suffers from h deprivations and individual j from l ($l < h$) deprivations. In distribution F_2, individual i suffers from $h + 1$ deprivations and individual j from $l - 1$ deprivations. The remaining individuals have identical status in F_1 and F_2. A social planner who supports the condition of second-degree downward count distribution dominance will consider F_1 to be preferable to F_2. By contrast, a social planner who supports the condition of second-degree upward count distribution dominance will prefer F_2 to F_1. Thus, for a fixed number of deprivations, second-degree downward dominance will rank the distribution with the lowest proportion suffering from all dimensions as more favourable then the distribution with the lowest proportion suffering from at least one dimension, whereas second-degree upward dominance provides a reverse ranking. Note that the criteria of second-degree downward and upward dominance are related to what Atkinson (2003) denotes the "intersection" and "union" approaches in multidimensional poverty assessment, which corresponds to the proportions of people suffering

from deprivation on all dimensions and those that suffer from at least one dimension. The normative justification of using either second-degree downward or upward dominance is discussed in Section 3.

2.2. Complete orderings — the dual approach

Since both second-degree downward and second-degree upward dominance in many cases will fail to provide complete rankings of deprivation count distributions, it will be helpful to introduce summary measures of deprivation.

Let $\boldsymbol{F}$ denote the family of deprivation count distributions. An ordering defined on $\boldsymbol{F}$ is a relation $\succcurlyeq$, which will be assumed to be continuous, transitive and complete and consequently can be represented by an increasing and continuous preference functional (see Debreu, 1964). To make the ordering relation $\succcurlyeq$ empirically relevant, we rely on the following independence condition[8]:

Axiom. *(Dual independence). Let F_1, F_2 and F_3 be members of $\boldsymbol{F}$ and let $\alpha \in [0,1]$. Then $F_1 \succcurlyeq F_2$ implies $(\alpha F_1^{-1} + (1 - \alpha) F_3^{-1})^{-1} \succcurlyeq (\alpha F_2^{-1} + (1 - \alpha) F_3^{-1})^{-1}$.*

This axiom requires that the ordering of distributions is invariant with respect to certain changes in the distributions being compared. If F_1 is weakly preferred to F_2, then the dual independence Axiom states that any mixture on F_1^{-1} is weakly preferred to the corresponding mixture on F_2^{-1}. The intuition is that identical mixing interventions on the inverse distribution functions being compared do not affect the ranking of distributions. Alternatively, one could invoke the primal independence axiom of Atkinson (1970), giving an expected utility representation of preferences. This axiom requires the preference ordering to be invariant with respect to identical mixing of the distribution functions being compared.

To illustrate the averaging operation associated with the dual independence Axiom, let us consider the problem of ranking distributions of couples obtained by matching men and women with the same rank in the male and female deprivation count distributions (i.e. the most deprived man is matched with the most deprived woman, the second deprived man with the second deprived woman, and so on). Dual independence means that, given any initial distribution F_3 of deprivation for the female population, if for the male population, distribution F_1 is deemed to contain less deprivation than distribution F_2, this judgement is not affected by the matching with female distribution F_3. The dual independence Axiom requires that this property holds regardless of the initial patterns of deprivation and of the weights associated to male and female deprivation counts when forming the couple distribution.

Theorem 2.1. *A preference relation $\succcurlyeq$ on $\boldsymbol{F}$ satisfies continuity, transitivity, completeness and dual independence if and only if there exists a continuous and non-decreasing real function Γ defined on the unit interval, such that for all F_1, $F_2 \in F$,*

$$F_1 \succcurlyeq F_2 \Longleftrightarrow \sum_{k=0}^{r-1} \Gamma(F_1(k)) \geq \sum_{k=0}^{r-1} \Gamma(F_2(k))$$

Moreover, Γ is unique up to a positive affine transformation.

Proof: See Appendix A.

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[7] Note that aggregating income distributions from above does not make sense since it conflicts with Pigou-Dalton's principle of transfers (see Aaberge, 2009).

[8] The dual independence Axiom was introduced by Yaari (1987) as an alternative to the independence axiom of the expected utility theory for choice under uncertainty. Weymark (1981) denoted this axiom Weak Independence of Income Source and used it to justify rank-dependent measures of inequality.

Theorem 2.1 provides a theoretical justification for the following family of social evaluation functions,

$$W_\Gamma(F) = \sum_{k=0}^{r-1} \Gamma(F(k)),$$ (2.3)

where Γ is a non-negative and non-decreasing continuous function that represents the preferences of the social planner, where the distribution that produces the largest $W_\Gamma(F)$ is the most favourable one. Thus, the social evaluation function $W_\Gamma(F)$ provides a normative justification of the following family of deprivation measures,[9]

$$D_\Gamma(F) = r - \sum_{k=0}^{r-1} \Gamma(F(k)),$$ (2.4)

where $\Gamma(0) = 0$ and $\Gamma(1) = 1$ for normalization purposes. Since F denotes the distribution of the deprivation count, $D_\Gamma(F)$ can be considered as a summary measure of deprivation exhibited by the distribution F. The social planner considers the distribution F that minimizes $D_\Gamma(F)$ to be the most favourable among those being compared, where $D_\Gamma(F) = 0$ if and only if $q_0 = 1$. The maximum value r for $D_\Gamma(F)$ is attained when $q_r = 1$. Comparing Eqs. (2.4) and (2.2), it follows that $D_\Gamma(F) = \mu$ when $\Gamma(t) = t$ and $\mu \leq D_\Gamma(F) \leq r$ when Γ *is convex*, and $0 \leq D_\Gamma(F) \leq \mu$ when Γ *is concave*. Notice that while income gives people consumption opportunities, deprivations are bad conditions that people would like to escape. Therefore, it makes sense to allow the preference function Γ of the social evaluation function defined by Eq. (2.4) to be convex as well as concave, whereas it is required to be concave when used as a welfare function for evaluating income distributions (consistent with Pigou-Dalton's principle of transfers). The convex and concave shape of Γ is associated with the distinction between the intersection and union approaches for measuring deprivation/poverty (see Atkinson et al., 2002 and Atkinson, 2003). An ethical view in favour of the union approach cares about the proportion of people who suffer from at least one dimension of deprivation $(1 - q_0)$, whereas the intersection approach focuses attention on the proportion of people deprived on all dimensions (q_r). By choosing

$$\Gamma(t) = \begin{cases} t & if \quad t \leq q_0 \\ 1 & if \quad q_0 < t \leq 1 \end{cases}$$ (2.5)

we get $D_\Gamma(F) = 1 - q_0$, which means that the proportion that suffers from at least one dimension can be considered as a limiting case of the D_Γ-family of measures of deprivation for concave Γ. The following alternative specification of the preference function,

$$\Gamma(t) = \begin{cases} 0 & if \quad t \leq 1 - q_r \\ t & if \quad 1 - q_r < t \leq 1 \end{cases}$$ (2.6)

yields $D_\Gamma(F) = r - 1 + q_r$ which means that the proportion that suffers from all dimensions represents a limiting case of the D_Γ-family of deprivation measures for convex Γ. Although the proportions suffering from at least one dimension and all dimensions do not belong to the D_Γ-family (which is generated by continuous Γ functions) these deprivation measures can be approximated within this class (see Le Breton and Peluso, 2010 for general approximation results).

2.3. Decomposition of the dual deprivation measures

As are well-known, the social welfare functions derived from the expected and rank-dependent utility theories, called primal and dual approaches below, allow for a multiplicative decomposition with respect

to the mean and the inequality of income distributions.[10] The deprivation measures defined by Eq. (2.4) are shown to possess a similar property by admitting an additive decomposition with respect to the mean and the dispersion of the deprivation count distributions. Since dispersion plays a crucial role in the decomposition of the deprivation measures it will be helpful to clarify what is meant by measures of dispersion. The standard measure of dispersion of a distribution function is the variance, which measures how far observations are spread out by the squared deviation of observations from the mean. Alternatively, a measure of dispersion can be derived from the variance of the empirical distribution function $F_n(x)$ (the non-parametric estimator of the cumulative distribution function), which is given by $var\sqrt{n}F_n(x) = F(x)(1 - F(x))$. Thus, the sum (integral) of $F(x)(1 - F(x))$ across the range of F emerges as an appropriate alternative to the variance as a measure of dispersion of the cumulative distribution function F. The measure $\int F(x)(1 - F(x))dx$ is called Gini's mean difference in the economic literature.[11] Gini's mean difference as well as the variance has symmetric properties in the sense that they treat a right skewed distribution and its left skewed mirror image as equally dispersed. However, when concern is turned to distributions that are either skewed to the left or to the right it will be useful to complement the information provided by the Gini's mean difference with measures of dispersion that are particularly sensitive to left- or right-spread tails.[12] To this end, we introduce the following family of dispersion measures,

$$\Delta_\Gamma(F) = \begin{cases} \sum_{k=0}^{r-1} [F(k) - \Gamma(F(k))] & \text{when } \Gamma \text{ is convex} \\ \sum_{k=0}^{r-1} [\Gamma(F(k)) - F(k)] & \text{when } \Gamma \text{ is concave}, \end{cases}$$ (2.7)

where $\Delta_\Gamma(F)$ can be considered as a right-spread measure of dispersion (or tail-heaviness) when Γ is convex and as a left-spread measure when Γ is concave. Inserting for the convex function $\Gamma(t) = t^2$ and the concave function, $\Gamma(t) = 2t - t^2$ in Eq. (2.7) yields Gini's mean difference (with negative sign in the concave case). Note that distributions that are skewed to the right (left) has a mean that typically is larger (smaller) than its median and are characterized by accumulation of observations towards the left (right) with a tail stretching towards the right (left). Distributions of income and wealth are typically skewed to the right.

Inserting Eq. (2.7) in Eq. (2.4) yields

$$D_\Gamma(F) = \begin{cases} \mu + \Delta_\Gamma(F) & \text{when } \Gamma \text{ is convex} \\ \mu - \Delta_\Gamma(F) & \text{when } \Gamma \text{ is concave} \end{cases}.$$ (2.8)

Thus, we may identify the contribution to D_Γ from the average number of deprivations μ and the dispersion of deprivations across the population. Expression (2.8) shows that a social planner with preference function $\Gamma(t) = t$ will only be concerned with reducing the mean number of deprivations, whereas a social planner who is also concerned with the dispersion of deprivations across the population will employ a measure D_Γ where Γ is either convex or concave. When Γ is convex, the social planner pays more attention to people who suffer from many deprivations than to people who suffer from few deprivations. By contrast, when the social planner uses the criterion D_Γ with a concave Γ, s/he is more concerned with the number of people who are deprived on one or more dimensions. Therefore, the dispersion measure is subtracted from the mean in the definition of the deprivation measure D_Γ for

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[9] It is shown in the appendix that the social welfare functions W_Γ and the associated D_Γ-measures satisfy the dual independence Axiom and fail to satisfy the primal independence axiom.

[10] See Atkinson's (1970) discussion of the equally distributed equivalent income for the primal case and Yaari (1988) for the dual case. Weymark (2006) discusses an extension to measurement of multidimensional inequality for the dual case.

[11] Gini's mean difference was originally introduced by von Andrae (1872) and Helmert (1876) as a more robust measure of dispersion than the variance.

[12] See e.g. Fernández-Ponce et al. (1998) and Shaked and Shanthikumar (1998) who provide a discussion on how to compare the right-spread variability of distribution functions.

concave Γ; the larger accumulation at the left tail the larger is the dispersion measure for concave Γ and the lower is the level of deprivation.

2.4. The Lorenz family of deprivation measures

To summarize the information of the location and the shape of a cumulative distribution function it is common to use the mean together with a few additional moments of the distribution function (second, third and fourth order moments, which provide information on spread, skewness and kurtosis). However, since a distribution function defined on the positive half line is uniquely determined by its mean and Lorenz curve, it is attractive to combine the mean with a few moments of the Lorenz curve. To this end, Aaberge (2000) introduced the Lorenz family of inequality measures defined by

$$J_i(L) = \frac{B_i(F)}{\mu} \tag{2.9}$$

and

$$B_i(F) = \int F(x)\left(1 - F^i(x)\right)dx, \quad i = 1, 2, \ldots, \tag{2.10}$$

where J_i was shown to be uniquely determined by the ith order moment of the Lorenz curve (L) associated with F.[13] Thus, since J_1, J_2 and J_3 are uniquely determined by the first, the second and the third moments of the Lorenz curve, they will jointly make up a fairly good summary of the Lorenz curve, which means that the mean μ and B_1, B_2, B_3 normally will provide a good description of the basic features of the distribution function F.[14] Now, by inserting the following specification for the preference function in Eq. (2.7),

$$\Gamma_i(t) = t^{i+1}, \tag{2.11}$$

we get

$$\Delta_\Gamma(F) = \Delta_{1i}(F) = \sum_{k=0}^{r-1} F(k)\left(1 - F^i(k)\right), \quad i = 1, 2, \ldots, \tag{2.12}$$

which demonstrates that $\Delta_{1i} = B_i$. By contrast, when the planner's preferences are consistent with a concave Γ then by inserting the following concave preference function

$$\Gamma_i(t) = 1 - (1 - t)^{i+1} \tag{2.13}$$

in Eq. (2.7), we get

$$\Delta_\Gamma(F) = \Delta_{2i}(F) = \sum_{k=0}^{r-1} (1 - F(k))\left(1 - (1 - F(k))^i\right), \quad i = 1, 2, \ldots. \tag{2.14}$$

Note that Δ_{1i} (Δ_{2i}) becomes more sensitive to changes that concern people that suffer from many (few) deprivations when $i \to \infty$. At the limiting case, $\mu + \Delta_{1i}$ and $\mu - \Delta_{2i}$ coincide with respectively the intersection and the union approach. For a further discussion of sensitivity to changes that concern the upper and lower tail of the count distribution, we refer to Section 3.

The two alternative quadratic specifications of Γ lead to the well-known Gini measure of dispersion $\Delta_1(F)$ and the associated Gini measure of deprivation $D_1(F)$.

$$\Delta_1(F) \equiv \Delta_{11}(F) = \Delta_{21}(F) = \sum_{k=0}^{r-1} F(k)(1 - F(k)) \tag{2.15}$$

and

$$D_1(F) = \begin{cases} \mu + \Delta_1(F) & \text{when } \Gamma(t) = t^2 \\ \mu - \Delta_1(F) & \text{when } \Gamma(t) = 2t - t^2. \end{cases} \tag{2.16}$$

It follows that Δ_1 is symmetric in the sense that it treats a right skewed distribution and its left skewed mirror image as equally dispersed. Note that Δ_{12} is particularly sensitive to changes that concern those people suffering from deprivation in many dimensions, whereas Δ_{22} is particularly sensitive to changes that concern those suffering from few dimensions. We refer to a further discussion of these properties in the next section. Thus, used together, Δ_1, Δ_{12} and Δ_{22} might give a good summary of the shape of the count distribution F and will be applied in Section 4 together with μ.

3. Normative justification of dominance criteria and deprivation measures

The axiomatic characterization of the family D_Γ of deprivation measures provides a normative justification of these measures. However, analogous to the role played by the Pigou-Dalton principle of transfers in measurement of income inequality, it is useful to introduce a normative principle that justifies employment of the deprivation measures D_Γ and the dominance criteria introduced in Section 2.1. To this end, the previous literature on measurement of multidimensional poverty and inequality in distributions of continuous variables has relied on the principle of correlation increasing transfers defined by Boland and Proschan (1988) and applied by e.g. Tsui (1999, 2002) and Alkire and Foster (2011), whereas Epstein and Tanny (1980) and Atkinson and Bourguignon (1982) provided an alternative definition in terms of correlation increasing perturbation which is particularly suitable for discrete distributions.[15] Both definitions are normally referred to as a correlation increasing rearrangement.

To illustrate the application of a correlation increasing rearrangement for distributions of deprivation counts, it will be helpful to consider the two-dimensional case. To this end, we start by clarifying the relationship between the joint distribution of the two deprivation dimensions X_1 and X_2, and the associated count distribution defined in Section 2.

Let $r = 2$, i.e. $X = X_1 + X_2$, $p_{ij} = \Pr((X_1 = i) \cap (X_2 = j))$, $p_{i+} = \Pr(X_1 = i)$, and $p_{+j} = \Pr(X_2 = j)$.

Thus, we get the following relationship between the count distribution parameters $q_k = \Pr(X = k)$, $k = 1, 2$ and the parameters p_{ij}, $i, j = 1, 2$ of the multinomial distribution of the two deprivation dimensions,

$$\begin{aligned} q_0 &= p_{00} \\ q_1 &= p_{10} + p_{01} \\ q_2 &= p_{11}. \end{aligned} \tag{3.1}$$

As illustrated by Table 3.1 a correlation increasing rearrangement (CIR) requires an equal increase in the number of people suffering from two dimensions and people that are not suffering from any dimension, and a corresponding reduction in the number of people suffering from dimension 1 and not from dimension 2 and in the number of people suffering from dimension 2 and not from dimension 1. The equal distribution of the reduction in the number of people suffering from one dimension is caused by the condition of fixed marginal distributions.

Definition 3.1. Consider a 2×2 table with parameters $(p_{00}, p_{01}, p_{10}, p_{11})$ where $\sum \sum p_{ij} = 1$. The following change $(p_{00} + \delta, p_{01} - \delta, p_{10} - \delta, p_{11}$

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[13] $\mu[(i + 1) \int u^i dL(u) - 1] = \int F(x)(1 - F^i(x))dx$

[14] We refer to Aaberge (2000) for a further discussion.

[15] For further discussion and application of association (correlation) increasing rearrangements under the condition of fixed marginal distributions, we refer to Dardanoni (1995), Tsui (1999, 2002), Bourguignon and Chakravarty (2003), Duclos et al. (2006), Weymark (2006) and Kakwani and Silber (2008). See also Tchen (1980) who deals with positive association (or concordance) between bivariate probability measures and Decancq (2012) for a recent generalization of these principles and an analysis of their links to stochastic dominance.

Table 3.1
The correlation increasing rearrangement.

	X_2			
		0	1	
X_1	0	$p_{00} + \delta$	$p_{01} - \delta$	p_{0+}
	1	$p_{10} - \delta$	$p_{11} + \delta$	p_{1+}
		p_{+0}	p_{+1}	1

Table 3.2
Illustration of a count neutral rearrangement.

	X_2			
		0	1	
X_1	0	p_{00}	$p_{01} - \gamma$	$p_{0+} - \gamma$
	1	$p_{10} + \gamma$	p_{11}	$p_{1+} + \gamma$
		$p_{+0} + \gamma$	$p_{+1} - \gamma$	1

$+ \delta$), with $\delta > 0$ ($\delta < 0$) is said to provide a correlation increasing (decreasing) rearrangement.

As indicated above, we are concerned with rearrangements that affect the count distribution (i.e. the parameters q_0, q_1 and q_2). Note, however, that the count distribution solely provides information on the number of deprivations, irrespective of whether they arise from dimension 1 or 2. To allow for mean preserving changes in the marginal distributions of deprivations, we introduce the following "count neutral rearrangements" (CNR).

Definition 3.2. Consider a 2×2 table with parameters $(p_{00}, p_{01}, p_{10}, p_{11})$ where $\sum \sum p_{ij} = 1$ The following change $(p_{00}, p_{01} - \gamma, p_{10} + \gamma, p_{11})$, where $\gamma \in [-1, 1]$ is said to provide a count neutral rearrangement.

The CNR principle is illustrated in Table 3.2, where the parameters of the multinomial distribution are affected by small amounts γ in such a way as to leave the deprivation count distribution unchanged, whereas the marginal distributions of X_1 and X_2 have changed.

The parameter γ only affects the allocation between the two dimensions (X_1 and X_2) of people that suffer from one dimension. Thus, CNR can be interpreted as a principle of neutrality of deprivation with respect to the different dimensions of deprivation.

The CNR rearrangement principle is crucial to understand the limits of the counting approach. By aggregating across deprivation variables, it is implicitly assumed that they are interpersonal comparable and can be summarized by a deprivation count distribution. The count neutral rearrangement principle elucidates the loss of information due to this aggregation process.

In the following subsection, we provide general results linking the two alternative rearrangement principles with dual deprivation measures, mean-preserving spread and dominance criteria.

3.1. Relationship between rearrangement principles, dominance criteria and deprivation measures

The following results provide characterizations of the relationship between second-degree downward and upward count distribution dominance and the general family D_Γ of deprivation measures. Moreover, Theorems 3.1A and 3.1B provide normative justification in terms of the two rearrangements principles presented above and of mean preserving spread/contractions, which are defined (on deprivation count distributions) by

Definition 3.3. Let F_1 and F_2 be members of the family F of count distributions based on r deprivation indicators and where F_1 and F_2 are assumed to have equal means. Then F_2 is said to differ from F_1 by a mean preserving spread (contraction) if $\Delta_\Gamma(F_2) > \Delta_\Gamma(F_1)$ for all convex Γ ($\Delta_\Gamma(F_2) < \Delta_\Gamma(F_1)$ for all concave Γ).

Note that Definition 3.3 is analogous to the mean preserving spread for continuous distributions introduced by Rothschild and Stiglitz (1970).

Next, let Ω_1 and Ω_2 be subfamilies of the family of Γ-functions introduced in Theorem 2.1, and defined by

$$\Omega_1 = \left\{ \Gamma : \Gamma'(t) > 0, \Gamma''(t) > 0 \text{ for all } t \in \langle 0, 1 \rangle, \text{ and } \Gamma'(0) = 0 \right\}$$

and

$$\Omega_2 = \left\{ \Gamma : \Gamma'(t) > 0, \Gamma''(t) < 0 \text{ for all } t \in \langle 0, 1 \rangle, \text{ and } \Gamma'(1) = 0 \right\}.$$

Note that $\Gamma'(0) = 0$ and $\Gamma'(1) = 0$ can be considered as normalization conditions.

We can now state the following theorem.

Theorem 3.1A. Let F_1 and F_2 be members of the family F of count distributions based on r deprivation indicators and assume that F_1 and F_2 have equal means. Then the following statements are equivalent

(i) F_1 second-degree downward (upward) dominates F_2.
(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ for all $\Gamma \in \Omega_1$ (for all $\Gamma \in \Omega_2$).
(iii) F_2 can be obtained from F_1 by a sequence of correlation increasing (decreasing) rearrangements and count neutral rearrangements.
(iv) F_2 can be obtained from F_1 by a mean preserving spread (contraction).

We refer to Appendix A.3 for a proof. Note that the equivalence between statements (i) and (ii) is true for all count distributions. Moreover, by adding the condition of elementary deprivation increases to the rearrangement principles of the previous theorem, we obtain Theorem 3.1B, which is a generalized version of Theorem 3.1A.

Definition 3.4. Let F_1 and F_2 be members of the family F of count distributions. Then F_2 is said to differ from F_1 by an elementary increase in deprivation if $F_1(i) > F_2(i)$ for any $i = 0, 1, 2, ..., r - 1$ and $F_1(j) = F_2(j)$ for all $j \neq i$.

Theorem 3.1B. Let F_1 and F_2 be members of the family F of count distributions based on r deprivation indicators with means μ_1 and μ_2, and assume that $\mu_1 \leq \mu_2$. Then the following statements are equivalent.

(i) F_1 second-degree downward (upward) dominates F_2.
(ii) $D_\Gamma(F_1) < D_\Gamma(F_2)$ for all $\Gamma \in \Omega_1$ (for all $\Gamma \in \Omega_2$).
(iii) F_2 can be obtained from F_1 by a sequence of correlation increasing (decreasing) rearrangements, count neutral rearrangements and/or elementary increases in deprivation.
(iv) F_2 can be obtained from F_1 by a mean preserving spread (contraction) and/or elementary increases in deprivation.

Proof: See Appendix A.4.

We will complete this subsection by a short discussion of how the Lorenz deprivation measures formed by the preference functions defined by Eqs. (2.11) and (2.13) respond to association rearrangements. To this end, we rely on Aaberge (2000), who evaluates the transfer sensitivity of rank-dependent measures of inequality based on Kolm's (1976) principle of diminishing transfers and the dual counterpart introduced by Mehran (1976). Both principles are used for unveiling the ethical properties of members of the Lorenz family of deprivation measures. The Lorenz deprivation measures defined by the preference functions $\Gamma_i(t) = t^{i+1}$ (defined by Eq. (2.11)) increase their sensitivity to

association rearrangements in the upper tail of the count distribution as i increases, i.e. the stronger convexity the more weight is placed on a correlation rearrangement that takes place in the upper part of the count deprivation, which corresponds to higher upside inequality aversion of social preferences. By contrast, the Lorenz deprivation measures defined by the preference function $\Gamma_i(t) = 1 - (1 - t)^{i+1}$ (defined by Eq. (2.13)) increase their sensitivity to rearrangements in the lower tail of the count distribution as i increases. At the limit, the measures associated with convex (concave) preference functions coincide with the intersection (union) approach in measurement of multidimensional poverty.

4. Changes in distributions of material deprivation in European countries during the Great Recession

This section applies the dual deprivation measures and the dominance results to assess the evolution of material deprivation in European countries during the Great Recession. Furthermore, we make an evaluation of whether the dual deprivation measures produce results that differ from the results obtained by using standard cut-off measures. To this end, we compare the EU countries (except Croatia) plus Norway, Switzerland and Iceland from 2005 to 2012 using the indicators of Material Deprivation (MD) collected by the EU Statistics on Income and Living Conditions (EU-SILC) project. The country-specific EU-SILC data sets contain between 7000 and 15,000 individuals above 16 years old.[16] Our unit of analysis is the individual, but we also attach household variables. The material deprivation indicators measure whether a person or household cannot afford:

1. to pay their mortgage or rent
2. to pay their utility bills
3. to keep their home adequately warm
4. to face unexpected expenses
5. to eat meat or proteins regularly
6. to go on holiday
7. a television set
8. a washing machine
9. a car
10. a telephone.

The individual is only considered to be suffering from deprivation on a specific dimension if he/she lacks the associated item because she cannot afford it. Non-response is treated as if the individual does not suffer from deprivation. There are very few individuals suffering from eight or more dimensions. Thus, to account for possible measurement errors in the proportions of individuals suffering from eight or more dimensions, such individuals are treated as suffering from seven dimensions.[17]

4.1. The impact of the Great Recession on deprivation in European countries

The impact of the Great Recession on material deprivation for 30 European countries is assessed on the basis of the Lorenz deprivation measures introduced in Section 2.4. In particular, we calculate the mean level of deprivation together with the dual deprivation measures with the convex preference functions $\Gamma(t) = t^{i+1}$, $i = 1, 2$ and the concave preference functions $\Gamma(t) = 1 - (1 - t)^{i+1}$, $i = 1, 2$. The full empirical results based on the five selected measures for each of the 30 countries between 2005 and 2012 are displayed in Fig. A.1 in the Online appendix. We present a summary below.

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[16] We use version 2005-3 from 01 to 03-08, version 2006-1 from 01 to 03-08, version 2007-2 from 01 to 08-09, version 2008-6 from 01 to 03-14, version 2009-6 from 01 to 03-14, version 2010-5 from 01 to 03-14, version 2011-3 from 01 to 03-14, and version 2012-1 from 01 to 03-14 which follow 30 countries (we exclude Croatia as only 2011 and 2012 is covered).

[17] This censoring of the data only affects the dominance results. Without the censoring, first order dominance and second order downward dominance are frequently violated, since the dominance criteria are very sensitive to the proportion suffering from the maximum numbers of dimensions.
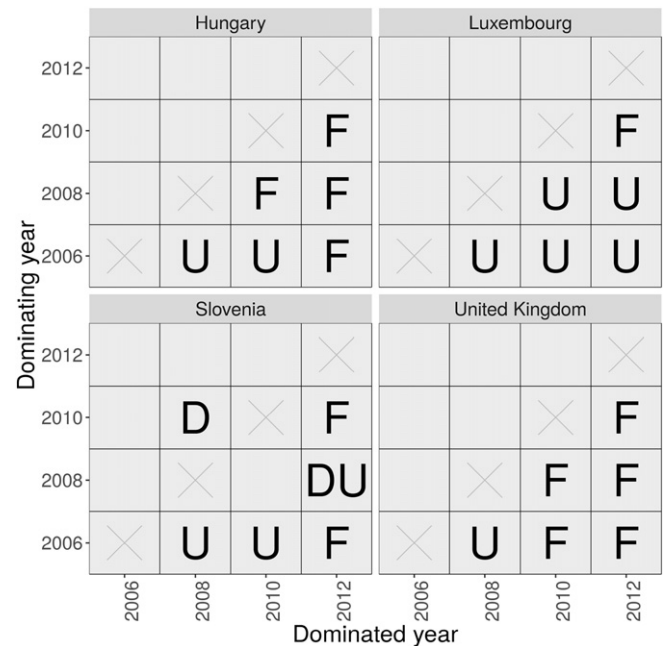

Fig. 4.1. Countries with robust trends of increasing deprivation. Note: The notation F, U and D means that the year in the row dominates the year in the column at the first order (F), at the second order upward (U) or at the second order downward (D).

The results show that Eastern European countries have the highest levels of material deprivation, but they have in general been less affected by the Great Recession than some Western European countries. While Hungary and Slovenia have experienced increased deprivation, Slovakia, Poland and Romania show decreasing deprivation trends over time.[18] A relatively stable pattern of deprivation was found in continental countries like France, Germany and Belgium, with a short-term stronger impact in Austria. By contrast, deprivation rose in UK, Ireland, Iceland, Luxembourg and the Netherlands, whereas Nordic countries as Finland, Norway and Sweden together with Switzerland were almost unaffected by the Great Recession. Finally, significant increases in material deprivation show to have taken place in the Mediterranean Countries Greece, Italy and Spain during the Great Recession.

To evaluate the robustness of the above results, we have used dominance criteria to make pairwise comparisons of count distributions in 2006, 2008, 2010 and 2012. Detailed results are provided by Fig. A.2 and discussed in the Online appendix. Fig. 4.1 shows the pattern of deprivation in terms of dominance criteria evaluations for a selected group of countries.

As demonstrated by Fig. 4.1, Luxembourg, UK, Slovenia and Hungary had all entries below the main diagonal filled by U, which mean that they experienced rising deprivation incidence over time for large families of deprivation measures. Notice that we have first order dominance of 2010 over 2012 both in Luxembourg, UK, and Slovenia uncovering a strong and persistent effect of the Great Recession on deprivation in these countries. It is worth noting that these results differ from those provided by Eurostat based on the MD rate, which find "Relatively Flat" material deprivation patterns for Luxembourg, UK and Slovenia.

4.2. Comparison with cut-off measures

It is common to use the proportion of individuals suffering in z or more dimensions for some cut-off z as a measure of deprivation (see e.g. Guio et al., 2017). An alternative approach is to use the dual

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[18] Note that our results for Czech Republic and Bulgaria differ from those of Guio et al. (2017). While their results show decreasing deprivation, our results reveal a U-shaped pattern.

Table 4.1
Kendall rank correlation between multidimensional deprivation measures.

Measure	Cut-off 2	Cut-off 3	Cut-off 4	$D_{1-(1-t)^3}$	$D_{1-(1-t)^2}$	D_t	D_{t^2}	D_{t^3}
Cut-off 2	1.00	0.71	0.53	0.84	0.86	0.87	0.84	0.76
Cut-off 3		1.00	0.69	0.66	0.69	0.77	0.79	0.82
Cut-off 4			1.00	0.51	0.54	0.63	0.67	0.71
$D_{1-(1-t)^3}$				1.00	0.96	0.87	0.81	0.72
$D_{1-(1-t)^2}$					1.00	0.90	0.85	0.76
D_t						1.00	0.94	0.86
D_{t^2}							1.00	0.92
D_{t^3}								1.00

Note: The cut-off z measure gives the proportion of people suffering from z or more dimensions. For each country a multidimensional deprivation measure generates a ranking of the years 2005–2012 showing which years had more material deprivation according to this measure. A cell in this table shows the Kendall rank correlation between the rankings generated by the column measure and the ranking generated by the row measure, averaged over all countries.

Table 4.2
Share of year pairs satisfying different dominance criteria.

Dominance criteria	Share of pairwise comparisons satisfying different dominance criteria (%)
First order	29
Second order upward	65
Second order downward	40
Second order downward and upward	36

Note: This table shows the proportion of year pairs within the same country where the distribution of multi-dimensional deprivation in one of the years is dominating the other year according to alternative dominance criteria.

measures of deprivation measures introduced in this paper. To assess to what extent different deprivation measures produce different conclusions regarding the evolution of material deprivation during the Great Recession, we have performed the following exercise: For a given country, each measure provides a ranking by deprivation for all years between 2005 and 2012. We then use Kendall's rank correlation coefficient to measure the association between the rankings given by two alternative deprivation measures. The averages across all countries of these rank correlations for selections of cut-off measures and deprivation measures are shown in Table 4.1.

Cut-off and dual deprivation measures give quite different conclusions regarding the development of material deprivation. For instance, the Kendall rank correlation between the mean deprivation and the proportion suffering from three or more dimensions is 0.77, which means that the two methods will produce different conclusions regarding whether material deprivation increased or decreased between two years for 11.5% of the pairwise comparisons.

4.3. Sensitivity of results to the specific choice of deprivation measure

An even more striking result of Table 4.1 is the sensitivity of conclusions with regard to the choice of specific cut-off measure. For instance, the Kendall rank correlation between the proportion suffering from more than one dimension and the proportion suffering more than three dimensions is merely 0.53, which means that the two cut-off measures produced different results for 23.5% of the pairwise comparisons.

Table 4.1 shows that conclusions are less sensitive to the choice of Γ function than to the choice of cut-off measure. For conclusions to be insensitive to the choice of cut-off threshold it is required that one distribution first-degree dominates the other. By contrast, for conclusions to be insensitive to the convex or concave specification of Γ for the deprivation measures it is sufficient that one distribution second-degree downward (upward) dominates the other. As shown by Table 4.2 there are only 29% of the pairwise comparisons that satisfy first-degree dominance, while there are 40% (65%) of the comparisons that satisfy second-degree downward (upward) dominance.

4.4. Illustration: Portugal between 2006 and 2012

To illustrate the importance of accounting for the information of the entire count deprivation, we consider the count distributions for Portugal in 2006 and 2012. The differences between the count distributions in 2012 and 2006 are displayed in Fig. 4.2. By comparing the proportions suffering from more than one dimension, Fig. 4.2 shows that 2012 exhibits lower deprivation than 2006, whereas comparisons of the proportions suffering from more than two dimensions shows lower deprivation in 2006. The reason is that there is a large decrease in the proportion of individuals suffering from two dimensions from 2006 to 2012, which compensates for the increase in the share of individuals suffering from three dimensions when one relies on cut-off measures. By contrast, using the methods introduced in this paper, one can easily verify that the 2012 distribution both upward and downward second-degree dominates the 2006 distribution. This means that all deprivation measures, irrespective of choice of convex or concave Γ, will state that the 2012 count distribution exhibits less material deprivation than the 2006 distribution.
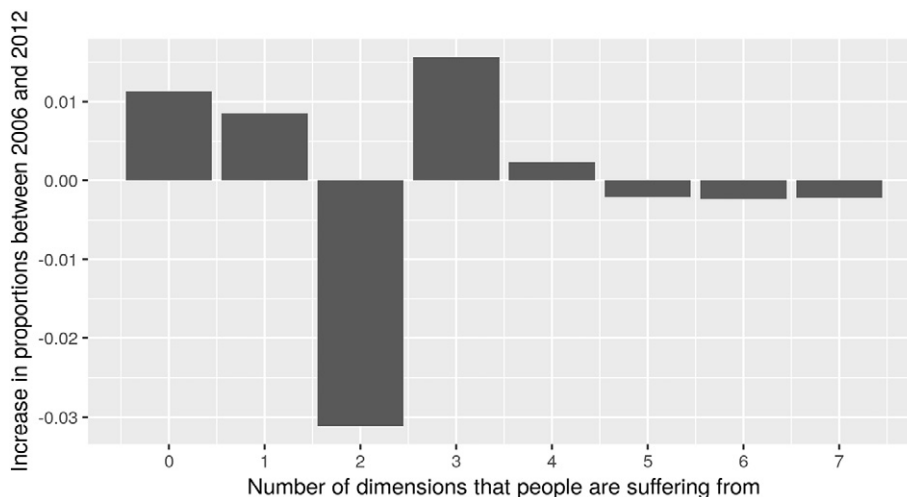
Fig. 4.2. The difference between deprivation count distributions in Portugal in 2012 and 2006. Note: This figure shows the difference between 2012 and 2006 in the proportion suffering from n dimensions, where n is ranging from 1 to 7 in Portugal. For instance, it shows that there were 3 percentage points fewer people suffering from two dimensions in 2012 than in 2006.

5. Summary and discussion

This paper introduces an axiomatically justified family of dual (rank-dependent) measures of multidimensional deprivation. These measures can be decomposed into the mean and the dispersion of deprivation counts, where the choice of the dispersion measure will depend on the social planner's concern for deprivation incidence versus deprivation severity. The normative properties of the deprivation measures can be judged by combinations of correlation increasing and count neutral rearrangements.

When applying the dual family of deprivation measures, we face the conventional "choice of measure" problem, since it for practical purposes normally will be convenient to restrict to a few measures of deprivation. To provide the practitioner with easily implementable and interpretable measures of multidimensional deprivation, we have introduced a subfamily of the dual measures called the Lorenz family of deprivation measures. The normative properties of the members of the Lorenz family depend on whether the associated preference function is convex or concave. Convexity (concavity) means that the social planner supports the principle of correlation increasing (decreasing) rearrangement. Moreover, as indicated in Section 3, Kolm's (1976) principle of diminishing transfers can be used to make further judgements of the normative properties of measures associated with convex (concave) preference functions, which provides helpful information for choosing a few complementary measures of deprivation for empirical work. However, a complete axiomatic characterization of each of these measures, similar as Aaberge (2001) did for the Gini coefficient, would nevertheless provide additional helpful information. We also see several other avenues for future research. First, while this paper has focused on material deprivation, the proposed methods can be applied in any setting where count data are available. See e.g. Olivera et al. (2018), who apply our methods to measure cognitive functioning inequality. Secondly, while it is straightforward to extend Theorem 2.1 to be valid for the case of weighted dimensions, it is more demanding to establish an analogous version of Theorems 3.1A and 3.1B for distributions of weighted count data. We leave this generalization for further research.

Acknowledgement

Appendix A. Proofs and extensions

A.1. Proof of Theorem 2.1

Since there is a one-to-one correspondence between the count distribution $\boldsymbol{F}$ and its inverse F^{-1}, we get that the ordering relation $\succeq$ defined on the set of inverse distribution functions is equivalent to the ordering relation defined on $\boldsymbol{F}$. Note that $F_1^{-1}(t) \leq F_2^{-1}(t)$ for all $t \in [0,1]$ if and only if $F_1(k) \geq F_2(k)$ for all $k = 0, 1, 2, ..., r-1$. Then, by replacing the primal independence axiom (defined on the set of distribution functions) with the dual independence Axiom (defined on the set of inverse distribution functions), Theorem 2 follows directly from the expected utility theorem, where $\Gamma(t)$ plays the role of the utility function and the ordering representation is given by.

$$\int_0^1 \Gamma(t)dF^{-1}(t) = \int_0^1 \Gamma(F(x))dx = \sum_{k=0}^{r-1} \Gamma(F(k)).$$

A.2. Independence axioms and dual measurement of deprivation

Following Yaari (1988) the dual welfare function for the distribution F of a variable that describes loss in well-being is defined by

$$W_\Gamma(F) = \int \Gamma(F(x))dx = \int \Gamma(t)dF^{-1}(t) \tag{A1}$$

To demonstrate that W_Γ satisfies the dual independence Axiom let us assume that F_1 and F_2 are such that $W_\Gamma(F_1) \geq W_\Gamma(F_2)$ for a non-negative and non-decreasing function Γ. By mixing the inverses of F_1 and F_2 with the inverse of an arbitrary third distribution F_3; i.e. F_i^{-1} is replaced by $\alpha F_i^{-1}(t) + (1-\alpha)F_3^{-1}(t)$ where $\alpha \in [0,1]$, $i = 1, 2$, we get that

$$W_\Gamma\left(\alpha F_1^{-1}(t) + (1-\alpha)F_3^{-1}(t)\right) - W_\Gamma\left(\alpha F_2^{-1}(t) + (1-\alpha)F_3^{-1}(t)\right) =$$

$$\int \Gamma(t)d\left[\left(\alpha F_1^{-1}(t) + (1-\alpha)F_3^{-1}(t)\right) - \left(\alpha F_2^{-1}(t) + (1-\alpha)F_3^{-1}(t)\right)\right] \tag{A2}$$

$$= \alpha \int \Gamma(t)d\left(F_1^{-1}(t) - F_2^{-1}(t)\right) = \alpha(W_\Gamma(F_1) - W_\Gamma(F_2)) \geq 0,$$

which shows that W_Γ satisfies the dual independence Axiom.

The primal independence axiom requires that social preferences are invariant with regard to mixing F_1 and F_2 with a third distribution F_3; i.e. preferences are not affected by replacing F_1 and F_2 by $\alpha F_1(t) + (1-\alpha)F_3(t)$ and $\alpha F_2(t) + (1-\alpha)F_3(t)$. However, $W_\Gamma(F_1) \geq W_\Gamma(F_2)$ does in general not imply that $W_\Gamma(\alpha F_1(t) + (1-\alpha)F_3(t)) \geq W_\Gamma(\alpha F_2(t) + (1-\alpha)F_3(t))$.

As an illustration, we will consider the following example. Let us consider three count distributions defined by their inverses:

$$F_1^{-1}(t) = \begin{cases} 0, & 0 < t \leq .5 \\ 2, & .5 < t \leq .9 \\ 4, & .9 < t \leq 1, \end{cases} \tag{A3}$$

$$F_2^{-1}(t) = \begin{cases} 0, & 0 < t \leq .6 \\ 2, & .6 < t \leq .8 \\ 4, & .8 < t \leq 1 \end{cases} \tag{A4}$$

and

$$F_3^{-1}(t) = \begin{cases} 0, & 0 < t \leq .5 \\ 4, & .5 < t \leq 1 \end{cases} \tag{A5}$$

and the following dual ($\tilde{F}_1$ and $\tilde{F}_2$) and primal (F_1^* and F_2^*) mixtures of F_1 and F_2 with F_3 where $\alpha = 0.5$.

Table A1
Illustrations of independence axioms.

Number of deprivations	0	1	2	3	4
F_1	0.5	0.5	0.9	0.9	1
F_2	0.6	0.6	0.8	0.8	1
F_3	0.5	0.5	0.5	0.5	1
$\tilde{F}_1$	0.5	0.5	0.5	0.9	1
$\tilde{F}_2$	0.5	0.5	0.6	0.8	1
F_1^*	0.5	0.5	0.7	0.7	1
F_2^*	0.55	0.55	0.65	0.65	1

Assume that $F_1 \succeq F_2$; i.e. $W_\Gamma(F_1) \geq W_\Gamma(F_2)$ for a non-negative and non-decreasing function $\Gamma \Leftrightarrow 2\Gamma(.5) + 2\Gamma(.9) \geq 2\Gamma(.6) + 2\Gamma(.8) \Leftrightarrow$

$$\Gamma(.9) - \Gamma(.8) \geq \Gamma(.6) - \Gamma(.5), \tag{A6}$$

which is equivalent to $W_\Gamma(\tilde{F}_1) \geq W_\Gamma(\tilde{F}_2)$.

Next, turning to the primal independence axiom, we find that $W_\Gamma(F_1^*) \geq W_\Gamma(F_2^*) \Leftrightarrow \Gamma(.70) - \Gamma(.65) \geq \Gamma(.55) - \Gamma(.50)$, which is not equivalent to Eq. (A6). This demonstrates that W_Γ does not satisfy the primal independence axiom.

A.3. Proof of Theorem 3.1A

To make the proof more transparent the two-dimensional case ($r = 2$) will be considered below. However, since intersections between distributions formed by r dimensions can be described by $r(r - 1)/2$ different 2×2 tables, the generalization to the r-dimensional case is straightforward. More precisely, since interventions affecting two specific dimensions are described by a two-dimensional table, involvement of several dimensions requires that the procedure demonstrated below for the two-dimensional case is carried out stepwise for the involved two-dimensional tables.

To simplify the proof, we relax the condition of fixed marginal distributions by combining CIR with count neutral rearrangement. The combined rearrangement is called "mean preserving association rearrangement". It is illustrated by Table A.2, where the parameters of the multinomial distribution are affected by small amounts δ and γ in such a way as to leave the mean number of deprivations unchanged. It follows from Table A.2 and Eq. (3.1) that

$$(p_{10} - \delta + \gamma) + (p_{10} - \delta - \gamma) + 2(p_{11} + \delta) = p_{10} + p_{10} + 2p_{11} = \ddot{E}$$

which means that the mean number of deprivations has not been affected by the intervention illustrated by Table 3.2, whereas the marginal distributions of X_1 and X_2 have changed when $\gamma \neq 0$.

Table A.2
Illustration of a mean preserving association rearrangement.

X_1	X_2			
		0	1	
	0	$p_{00} + \delta$	$p_{01} - \delta - \gamma$	$p_{0+} - \gamma$
	1	$p_{10} - \delta + \gamma$	$p_{11} + \delta$	$p_{1+} + \gamma$
		$p_{+0} + \gamma$	$p_{+1} - \gamma$	1

Definition 3.1. Consider a 2×2 table with parameters $(p_{00}, p_{01}, p_{10}, p_{11})$ where $\sum \sum p_{ij} = 1$. The following change $(p_{00} + \delta, p_{01} - \delta - \gamma, p_{10} - \delta + \gamma, p_{11} + \delta)$ is said to provide a mean preserving association increasing (decreasing) rearrangement if $\delta > 0$ ($\delta < 0$) where $\gamma \in [-1, 1.]$

By applying Definition 3.1, we get that statement (iii) is equivalent to the following statement

(iii)* F_2 can be obtained from F_1 by a sequence of mean preserving association increasing (decreasing) rearrangements.

The principle of correlation increasing rearrangement can be considered as a special case of the principle of mean preserving association increasing rearrangement. In this case the reduction (2δ) in the proportion of those suffering from one deprivation is equally allocated between the two indicators X_1 and X_2. When $\gamma = \delta$ or $\gamma = -\delta$ the proportion suffering from either dimension 1 or from dimension 2 is reduced by 2δ. This case has been considered by Aaberge and Peluso (2012) and Aaberge and Brandolini (2015). Similarly, the count neutral rearrangement principle coincides with the special case where $\delta = 0$.

We begin by proving the equivalence between statements (i) and (iii).

Let $F_1(k) = \sum_{j=0}^{k} q_{1j}$ and $F_2(k) = \sum_{j=0}^{k} q_{2j}$, $k = 0, 1, 2$

where q_{ij} is the proportion suffering from j dimensions in distribution i. By inserting for F_1 and F_2 in Definition 2.1A, we get that F_1 second-degree downward dominates F_2 if and only if

$$\sum_{k=i}^{1} \sum_{j=0}^{k} q_{1j} \geq \sum_{k=i}^{1} \sum_{j=0}^{k} q_{2j} \text{ for } i = 0, 1. \tag{A10}$$

Let θ_j be defined by $\theta_j = q_{2j} - q_{1j}$, $j = 0, 1$. Then it follows that the distance between F_2 and F_1 can be described by two parameters, θ_0 and θ_1, i.e.

$$F_2(k) - F_1(k) = \sum_{j=0}^{k} q_{2j} - \sum_{j=0}^{k} q_{1j}$$
$$= \begin{cases} q_{20} - q_{10} = \theta_0, & k = 0 \\ q_{20} - q_{10} + q_{21} - q_{11} = \theta_0 + \theta_1, & k = 1 \\ 1 - 1 = 0, & k = 2. \end{cases} \tag{A11}$$

The condition of fixed mean requires that

$$0 = q_{21} + 2q_{22} - q_{11} - 2q_{12} = q_{11} + \theta_1 + 2(1 - q_{11} - \theta_1 - q_{10} - \theta_0)$$
$$- q_{11} - 2(1 - q_{11} - q_{10}) = -2\theta_0 - \theta_1,$$

which implies that $\theta_1 = -2\theta_0$. Inserting for $\theta_1 = -2\theta_0$ in Eq. (A11) yields

$$F_2(k) - F_1(k) = \sum_{j=0}^{k} q_{2j} - \sum_{j=0}^{k} q_{1j} = \begin{cases} \theta_0, & k = 0 \\ -\theta_0, & k = 1 \\ 0, & k = 2, \end{cases} \tag{A12}$$

which implies that

$$\sum_{k=i}^{1} F_1(k) - \sum_{k=i}^{1} F_2(k)$$
$$= \begin{cases} F_1(0) - F_2(0) + F_1(1) - F_2(1) = -\theta_0 + \theta_0 = 0, & i = 0 \\ F_1(1) - F_2(1) = \theta_0, & i = 1. \end{cases} \tag{A13}$$

Next, assume that F_1 is affected by an increasing association rearrangement. Thus, it follows from Definition 3.1 (and Eq. (A12)) that the distance between the resulting distribution F^* and F_1 is given by

$$F^*(k) - F_1(k) = \begin{cases} \delta, & k = 0 \\ -\delta, & k = 1 \\ 0, & k = 2, \end{cases} \tag{A14}$$

which means that the distance of the aggregated distributions is given by

$$\sum_{k=i}^{1} F_1(k) - \sum_{k=i}^{1} F^*(k) = \begin{cases} 0, & i = 0 \\ \delta, & i = 1. \end{cases} \tag{A15}$$

When $\delta > 0$, it follows from Eqs. (A15) and (A13) by choosing $F_2(k) = F^*(k)$ that (iii) is equivalent to (i).

Next, we will prove the equivalence between (i) and (iv). As was demonstrated above the distance between two distributions F_2 and F_1 with equal mean can be described by Eq. (A12). Inserting for Eq. (A12) in Eq. (2.7) when $r = 2$ yields

$$\Delta_\Gamma(F_2) - \Delta_\Gamma(F_1) = (\Gamma(q_{10} + q_{11}) - \Gamma(q_{10} + q_{11} - \theta_0)) - (\Gamma(q_{10} + \theta_0) - \Gamma(q_{10})). \tag{A16}$$

It follows from Eq. (A10) and the definition of convexity that $\Delta_\Gamma(F_2) - \Delta_\Gamma(F_1) > 0$ for a (non-decreasing) convex function $\Gamma(t)$ if and only if $\theta_0 > 0$, which according to Eq. (A13) means that F_1 second-degree downward dominates F_2.

What remains to be proved is the equivalence between (ii) and (iv), which follows directly from the decomposition Eq. (2.8).

The proof for the concave case has been omitted since it is analogous to the proof for the convex case.□

To prove Theorem 3.1B, it is helpful to introduce the following definition and lemma.

Definition A1. Let F be a count distribution. A lower (upper) elementary deprivation increase is a decrease in $F(j)$ where j is the lowest integer with $F(j) > 0$ ($F(j) = 1$) and $F(i)$ is kept unchanged for $i \neq j$.

Lemma A.1. Assume F_1 second-degree downward (upward) dominates F_2 where $\mu_1 < \mu_2$. Let F_1^* with mean μ_1^* differ from F_1 by a lower (upper) elementary deprivation increase with $\mu_1^* \leq \mu_2$. Then F_1^* weakly second-degree downward (upward) dominates F_2.

Proof. Assume F_1 second-degree downward dominates F_2. Then we need to show that

$\sum_{k=i}^{r-1} F_1^*(k) \geq \sum_{k=i}^{r-1} F_2(k)$ $for\, i = 0,1,2,...,r-1$. This is trivially true for $i > j$, where j is the lowest integer with $F(j) > 0$. For $i \leq j$, we get from Eq. (2.2) that

$\sum_{k=i}^{r-1} F_1^*(k) = r - \mu_1^* \geq r - \mu_2 = \sum_{k=i}^{r-1} F_2(k)$ $for\, i = 0,1,2,...,r-1$.

The proof for upward dominance is analogous to the downward dominance case. □

A.4. Proof of Theorem 3.1B

As for the proof of Theorem 3.1A, we consider the two-dimensional distribution and omit the concave case since it is analogous to the proof for the convex case. We start by proving the equivalence between (i) and (ii).

Let $F_s(k) = \sum_{j=0}^{i} q_{sj}$, $i = 0,1,2$; $s = 1,2$, where $\mu_s = q_{s1} + 2q_{s2}$ is the mean of F_s. Note that statement (i) is given by

(I) $q_{10} + q_{11} \geq q_{20} + q_{21}$

(II) $2q_{10} + q_{11} \geq 2q_{20} + q_{21}$ which is equivalent to $q_{10} + q_{11} - (q_{20} + q_{21}) \geq q_{20} - q_0$,

and that condition (II) is equivalent to $\mu_1 \leq \mu_2$, which follows from Eq. (2.2).

Next, assume that statement (i) of Theorem 3.1B is true

$\Leftrightarrow$

$D_\Gamma(F_2) - D_\Gamma(F_1) \geq 0$ for all $\Gamma \in \Omega_1$

$\Leftrightarrow$

$\Gamma(q_{10} + q_{11}) - \Gamma(q_{20} + q_{21}) - (\Gamma(q_{20}) - \Gamma(q_{10})) \geq 0$ for all non-decreasing convex Γ

$\Leftrightarrow$

$\Gamma(q_{10} + q_{11}) - \Gamma(q_{20} + q_{21}) \geq \Gamma(q_{20}) - \Gamma(q_{10})$ for all non-decreasing convex Γ

$\Leftrightarrow$

$q_{10} + q_{11} \geq q_{20} + q_{21}$

since $q_{10} + q_{11} - (q_{20} + q_{21}) \geq q_{20} - q_{10}$

$\Leftrightarrow$

$\sum_{k=i}^{1} F_1(k) \geq \sum_{k=i}^{1} F_2(k)$, $i = 0,1$.

Next, we will prove the equivalence between (i), (iii) and (iv). Proof that (i) implies (iii) and (iv). Assume that F_1 second-degree downward dominates F_2. (The proof for upward dominance is similar.) Let F_1^* (with mean μ_1^*) be derived from F_1 by applying lower elementary deprivation increases until $\mu_1^* = \mu_1$. By Lemma A.1, we get that F_1^* weakly second-degree downward dominates F_2. Thus, by Theorem 3.1A, F_2 can be obtained from F_1^* by a sequence of mean preserving association increasing rearrangements (iii), and F_2 can be obtained from F_1^* by a mean preserving spread (iv).

Proof that (iii) and (iv) implies (i). By Theorem 3.1A, any mean preserving association increasing (decreasing) rearrangement and mean preserving spread (contraction) leads the resulting distribution to

second-degree downward (upward) dominate the original distribution. And any elementary deprivation increase leads the resulting distribution to first order dominate the original distribution. □

A.5. Extension to higher dimensions

Definition 3.1 can readily be extended to higher dimensions. However, in cases of many dimensions the standard subscript notation becomes cumbersome. Thus, we find it convenient to introduce the following simplified subscript notation p_{ijm}, where i and j represents the outcomes 0 and 1 of two arbitrary chosen deprivation dimensions and m represents the remaining $r-2$ dimensions, where m is a $(r-2)$-dimensional vector of any combination of zeroes and ones.

To deal with r-dimensional counting data, we introduce the following generalization of Definition 3.1,

Definition 3.2. Consider a $2 \times 2 \times ... \times 2$ table formed by r dichotomous variables with parameters $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$. The following change $(p_{iim} + \delta, p_{ijm} - \gamma, p_{jim} - 2\delta + \gamma, p_{jjm} + \delta)$ is said to provide a mean preserving association increasing (decreasing) rearrangement if $\delta > 0$ ($\delta < 0$) where $\gamma \in [0,1]$.

Appendix B. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.jpubeco.2019.06.004.

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