

Ranking intersecting distribution functions

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Summary

Second-degree dominance has become a widely accepted criterion for ordering distribution functions according to social welfare. However, it provides only a partial ordering, and it may fail to rank distributions that intersect. To rank intersecting distribution functions, we propose a general approach based on rank-dependent theory. This approach avoids making arbitrary restrictions or parametric assumptions about social welfare functions and allows researchers to identify the weakest set of assumptions needed to rank distributions according to social welfare. Our approach is based on two complementary sequences of nested dominance criteria. The first (second) sequence extends second-degree stochastic dominance by placing more emphasis on differences that occur in the lower (upper) part of the distribution. The sequences characterize two separate systems of nested subfamilies of rank-dependent social welfare functions. This allows us to identify the least restrictive rank-dependent social preferences that give an unambiguous ranking of a given set of distribution functions. We also provide an axiomatization of the sequences of dominance criteria and the corresponding subfamilies of social welfare functions. We show the usefulness of our approach using two empirical applications; the first assesses the welfare implications of changes in household income distributions over the business cycle, while the second performs a social welfare comparison of the actual and counterfactual outcome distributions from a policy experiment.

KEYWORDS

distribution functions, inequality, social welfare, stochastic dominance

1 | INTRODUCTION

How do we compare intersecting distribution functions? The answer to this question is important for both descriptive analysis and policy evaluation. A key task of statistical offices and government agencies is to compare distributions of economic variables across countries, subgroups, and time. Much descriptive research is about analyzing changes in or differences between distributions of wages, income, consumption, and wealth, as they are considered important determinants of economic welfare as well as markers for what kind of activities are rewarded in an economy. There is also a growing body of research on how to assess the distributional effects of policy changes: the literature has developed methods for estimating the counterfactual outcome distribution in the absence of a policy intervention but has generally stopped short of establishing a framework for ranking the actual and counterfactual outcome distributions.

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In this paper, we develop a rank-dependent theory for ordering distribution functions according to social welfare. Since the seminal contributions of Hardy et al. (1934), Kolm (1969), and Atkinson (1970), second-degree stochastic dominance has become a widely accepted criterion for ranking distribution functions. But in many applications where the distribution functions intersect, a reasonable refinement of this criterion is necessary to attain an unambiguous ranking.¹ Although the theoretical literature offers dominance criteria of third or higher degree,² they are rarely used. Atkinson (2008) points out why higher degree dominance criteria are difficult to interpret and hard to justify because they rely on assumptions about third or higher order derivatives of the utility function. Thus, most empirical studies consider a few moments of the distribution function or use a parametric social welfare function when ranking intersecting distribution functions. The challenge in ranking distribution functions by their moments is twofold. First, the moments of an unbounded distribution do not uniquely determine the distribution function. For example, there exists several distributions with the same moments as the log-normal distribution (Heyde, 1963). Second, it is not clear how to aggregate and weigh the various moments of the distributions being compared. A natural concern is that the conclusions reached in these studies are sensitive to the choice of moments or to the specification of the social welfare function.

Our approach for comparing intersecting distribution functions is based on two complementary sequences of nested inverse stochastic dominance criteria.³ The first sequence includes the traditional inverse dominance criteria of third and higher degrees; it is called *upward dominance* because it aggregates the inverse of the distribution function from below and therefore places more emphasis on differences that occur in the lower part of the distribution. The second sequence complements the upward dominance criteria by placing more emphasis on differences that occur in the upper part of the distribution; we call it *downward dominance* because it aggregates the integrated inverse distribution function from above. Since the sequences are hierarchical, the sensitivity to differences in the lower (upper) part of the distribution increases with the degree of upward (downward) dominance. The two sequences coincide at second-degree dominance, and thus both satisfy the Pigou–Dalton transfer principle.

For each sequence, we show that inverse stochastic dominance of any degree can be given a simple social welfare interpretation. For example, ranking distribution functions according to third-degree upward inverse stochastic dominance is equivalent to using a version of the Gini social welfare function that compares the welfare of individuals below every quantile of the distributions in question.⁴ As a consequence, it is not necessary to rely on assumptions about third and higher order derivatives to interpret the sequences of dominance criteria. In the limit, the two sequences can be interpreted as polar alternatives for ordering distribution functions. The highest degree of downside inequality aversion is achieved when focus is exclusively turned to the situation of the poorest in the population. In this case, the social welfare function corresponds to the Rawlsian maximincriterion. By contrast, the highest degree of upside inequality aversion is achieved when focus is exclusively turned to the mean income. In this case, the social welfare function corresponds to the mean income criterion.

We next characterize the relation between criteria of upward and downward dominance and social welfare functions. For each sequence, we show equivalence in the ranking of distributions according to the dominance criteria and a general family of rank-dependent social welfare functions. The family of rank-dependent social welfare functions was originally proposed by Weymark (1981) and Yaari (1987, 1988) and can be represented as weighted averages of the ordered outcomes of interest where the weight decreases with the rank in the outcome distribution. The functional form of the weighting function details the inequality aversion of a social planner who employs the family of social welfare functions to compare intersecting distribution functions. Because the sequences of dominance criteria are nested, our equivalence results allow us to uniquely identify the largest subfamily of rank-dependent welfare functions—and thus the least restrictive rank-dependent social preferences—that give an unambiguous ranking of a given set of distribution functions.

We also provide an axiomatic characterization of the largest subfamily of social welfare functions that rank consistently with dominance of any given degree. Because of the equivalence result, this characterization gives a normative justification not only for the social welfare functions but also for the use of higher degree dominance criteria when comparing distribution functions. The subfamily associated with upward dominance is characterized by (generalizations of) the

¹Several studies have demonstrated the limited practical scope for ranking income distributions according to second-degree stochastic dominance (see, e.g., Davies & Hoy, 1995, and Atkinson, 2008)

²See, for example, Fishburn (1976), Fishburn (1980), Chew (1983), and Fishburn and Willig (1984) for extensions of stochastic dominance to an arbitrary order.

³While second-degree inverse stochastic dominance is equivalent to second-degree stochastic dominance (Hardy et al., 1934; Kolm, 1969; Atkinson, 1970), the two types of dominance differ at the third or higher degree. See, for example, Le Breton and Peluso (2009) for a discussion.

⁴The Gini social welfare function was originally introduced by Sen (1974) and was given a complete axiomatic justification by Aaberge (2001).

principle of downside positional transfer sensitivity (DPTS) (see Zoli, 1999, and Aaberge, 2000, 2009), whereas the subfamily associated with downward dominance is characterized by (generalizations of) the principle of upside positional transfer sensitivity (UPTS) (see Aaberge, 2009). The two principles differ in the sensitivity to differences in the lower versus upper part of the distribution.

To not only answer whether one distribution is better than another distribution but also get an estimate of by how much, it is necessary to work with parametric social welfare functions. We show that the members of two alternative parametric families of social welfare functions can be divided into subfamilies according to their relationship with the nested inverse stochastic dominance criteria. The parametric family that ranks consistently with upward (downward) dominance criteria exhibits successively higher aversion to differences in the lower (upper) part of the distribution. The parametric families are well known, and each of them has been shown to uniquely determine the distribution function; that is, no information is lost by restricting focus to these parametric social welfare functions.

We show the usefulness of our approach using two empirical applications. The first application uses data from the United Kingdom to study how the distribution of household income evolved over a boom and a bust era in the British economy. We show how our approach can be used to make unambiguous statements about the social welfare implications of the changes in the household income distribution over the business cycle. The second application uses random-assignment data to evaluate the distributional effects of Connecticut's Jobs First program, which involved generous earnings disregard and strict time limits.⁵ We use our approach to infer the least restrictive social preferences that allow an unambiguous conclusion of whether this program was an overall success. In both applications, we find that third-degree downward dominance is a particularly powerful refinement of second-degree dominance, providing an almost complete ranking of the distribution functions. By comparison, the traditional criterion of third-degree upward dominance resolves few of the comparisons that were ambiguous under second-degree dominance.

The remainder of the paper proceeds as follows. The next subsection discusses how our paper relates and contributes to the literature. Section 2 characterizes the relationship between inverse stochastic dominance and social welfare functions as criteria for ranking distribution functions. Section 3 identifies and describes the parametric families that rank distributions consistent with upward and downward dominance. Section 4 presents our two empirical applications before Section 5 offers some concluding remarks.

1.1 | Relation and contribution to the literature

Our paper is primarily related to a growing literature on refinements of second-degree dominance in the comparison of distribution functions (for reviews, see Lambert, 1993, and Le Breton & Peluso, 2009). There are two alternative approaches to introduce and justify such refinements. One alternative is to use an expected utility representation of social preferences, which offers higher degree stochastic dominance criteria. This approach corresponds to imposing conditions on the derivatives of the utility function. The other alternative is to use a rank-dependent representation of social preferences, which is associated with higher-degree *inverse* stochastic dominance criteria. This approach corresponds to imposing conditions on the derivatives of a weighting function.⁶ The rank-dependent representation of preferences has gained popularity in several other areas of economics. For example, rank-dependent theory has proven useful in resolving important paradoxes in choice under uncertainty. Additionally, rank-dependent theory has become a work horse for measurement of inequality, in part because of its direct link to Lorenz dominance, but also due to the frequent use of the Gini coefficient.

In this paper, we follow the rank-dependent approach. Condition on the rank-dependent representation of preferences, we are able to identify the least restrictive social preferences that give an unambiguous ranking of any set of distribution functions. A natural question is what are the key differences between approaches based on rank-dependent and expected utility representation of social preferences. Theoretically, the difference lies in the choice of independence axiom. By invoking the dual independence axiom, we restrict attention to rank-dependent social preferences. The dual independence axiom requires the preference ordering to be invariant with respect to identical mixing of the *inverses of the distribution functions* being compared, that is, mixing of income levels given population shares. Alternatively, one could

⁵Our choice to use the Jobs First program is not incidental: As shown in Bitler et al. (2006), the estimated quantile treatment effects exhibit the substantial heterogeneity predicted by the labor supply theory. As a consequence, the distributions of income with and without the Jobs First program intersect.

⁶Although second-degree inverse stochastic dominance is equivalent to second-degree stochastic dominance, the two types of dominance differ at the third or higher degree.

invoke the independence axiom of Atkinson (1970), giving an expected utility representation of preferences. This axiom requires the preference ordering to be invariant with respect to identical mixing of the *distributions* being compared, that is, mixing of population shares given income levels. While we refer to Weymark (1981), Yaari (1988), and Aaberge (2001) for a detailed discussion of these independence axioms, a simple example may be useful to appreciate the key differences. Consider a population with two groups, the rich and the poor. In each group, every member has the same income, the group-specific mean. By invoking the dual independence axiom, one is more concerned with the number of poor rather than how poor they are. By comparison, Atkinson's independence axiom focuses attention on the income share of the poor as opposed to how many they are.

Another difference between the two approaches is that the rank-dependent framework has certain analytical advantages, partly because one is working with the inverse of the distribution function that is defined on a bounded interval. First, it allows us to interpret inverse stochastic dominance criteria in terms of equally distributed equivalent incomes (see Propositions 2.1–2.4). To the best of our knowledge, stochastic dominance criteria cannot be given such an intuitive economic interpretation. Second, the rank-dependent framework allows us to derive two complementary sequences of upward and downward dominance criteria and explore their relationship to social welfare functions (see Theorems 2.2 and 2.4). We are not aware of results for downward dominance criteria within the expected-utility framework. Third, we can achieve equivalence results between dominance criteria, transfer principles, and social welfare functions without imposing restrictions on the distribution functions being compared (see Theorems 2.1 and 2.3). This allows us to identify the least restrictive rank-dependent social preferences that give an unambiguous ranking of a given set of distribution functions. This contrasts with the results of Whitmore (1970), Chew (1983), Shorrocks and Foster (1987), and Davies and Hoy (1995).⁷ These authors rely on an expected utility representation of preferences, and their dominance results require imposing restrictions on the distribution functions that are being compared. For example, Chew (1983) relies on a definition of *k*th-degree stochastic dominance where the *k* first moments of the distributions being compared are equal. In contrast, our approach allows any relationship between these moments. Building on Shorrocks and Foster (1987), Davies and Hoy (1995) show that third-degree stochastic dominance amounts to comparisons of the coefficient of variation for every cumulative “subpopulation” defined by a point of intersection between the Lorenz curves. Their result, however, assumes that the distributions being compared have the same means. Alternatively, one may use their results to study the distribution of relative incomes, performing inequality comparisons but not ranking distributions according to social welfare.

Our paper is also related to a literature on how to draw inference about inequality by ranking Lorenz curves in situations where these curves intersect and no unambiguous ranking can be attained without introducing weaker ranking criteria than first-degree Lorenz dominance. Muliere and Scarsini (1989) and Zoli (1999) examine the implications of applying second-degree Lorenz dominance as a criterion for ranking Lorenz curves. Zoli (2002) investigates the relationship between the third-degree inverse stochastic dominance and inequality dominance when Lorenz curves intersect. He also explores the link between higher degrees of upward inverse dominance and social welfare as measured by a parametrized family of extended Gini indices. Aaberge (2009) continues this exploration, characterizing the relationship between various Lorenz dominance criteria and a general family of rank-dependent measures of inequality. He proposes two alternative sequences of nested dominance criteria. One sequence focuses on the lower part of the Lorenz curves (upward Lorenz dominance), whereas the other focuses on the upper part of the Lorenz curve (downward Lorenz dominance). Insofar as the distributions being compared have equal means, his results are also informative about the relation between upward and downward inverse stochastic dominance and rank-dependent measures of social welfare. If the distributions have different means, however, there are no direct implications from the results presented in Aaberge (2009).

2 | INVERSE STOCHASTIC DOMINANCE AND SOCIAL WELFARE

This section begins by reviewing the relationship between second-degree dominance and rank-dependent social welfare functions. We next introduce upward and downward dominance of third degree as criteria for ranking distribution functions and characterize their relationship to social welfare functions. Finally, we introduce the full hierarchical sequences of nested inverse stochastic dominance criteria and show how they allow us to uniquely identify the largest subfamily of social welfare functions required to reach an unambiguous ranking of a given set of distribution functions.

⁷See also Dardanoni and Lambert (1988) and Davies and Hoy (1994).

2.1 | Second-degree dominance and rank-dependent welfare functions

Let F be a member of the set \mathcal{F} of cumulative distribution functions with mean μ_F and left inverse defined by

$$F^{-1}(t) = \inf\{x : F(x) \geq t\}.$$

Note that both discrete and continuous distribution functions are allowed in \mathcal{F} , and though the former is what we actually observe, the latter often allows simpler derivation of theoretical results and is a valid large sample approximation. Thus, in most cases below, F will be assumed to be a continuous distribution function, but the assumption of a discrete distribution function will be used where appropriate. To fix ideas, we will refer to F as the income distribution, although our framework can be applied to any type of distribution function.

2.1.1 | Second-degree dominance

Since the seminal contributions of Kolm (1969) and Atkinson (1970), second-degree dominance has become a widely accepted criterion for ranking distribution functions.⁸

Definition 2.1. A distribution function F_1 is said to second-degree dominate a distribution function F_0 if and only if

$$\int_0^u F_1^{-1}(t)dt \geq \int_0^u F_0^{-1}(t)dt \text{ for all } u \in [0, 1]$$

and the inequality holds strictly for some $u \in (0, 1)$.

As is well known, all inequality averse social planners rank distribution functions consistently with second-degree dominance. But in many applications, weaker criteria than second-degree dominance are required to obtain an ordering of distributions.

2.1.2 | Rank-dependent social welfare functions

As in the literature on choice under uncertainty, ranking criteria can be derived from independence axioms imposed on the ordering \succsim defined on \mathcal{F} . The preference relation \succsim of the social planner is assumed to be continuous, transitive, and complete and to rank $F_1 \succsim F_0$ if $F_1^{-1}(t) \geq F_0^{-1}(t)$ for all $t \in [0, 1]$. To give the preferences of the planner an empirical content, Yaari (1987, 1988) imposes the so-called dual independence axiom on \succsim , defined by

Axiom (Dual independence). Let F_0 , F_1 , and F_2 be members of \mathcal{F} and let $\alpha \in [0, 1]$. Then $F_1 \succsim F_0$ implies $(\alpha F_1^{-1} + (1 - \alpha)F_2^{-1})^{-1} \succsim (\alpha F_0^{-1} + (1 - \alpha)F_2^{-1})^{-1}$.

Armed with this axiom, Yaari ((1987), (1988)) proved that the preference relation \succsim can be represented by the following rank-dependent family of social welfare functions⁹

$$W_P(F) = \int_0^1 P'(t)F^{-1}(t)dt, \quad (2.1)$$

where $P'(t) > 0$.

Note that the functions W_P are equally valid as aggregate measures of social welfare if the social planner is concerned with the distribution of some transformation of the incomes (such as individual utilities) rather than the distribution of incomes (or some other economic outcome such as wealth or education). In our analysis, we use income as the running example for the argument of the social welfare function, such that $F^{-1}(t)$ is the t quantile of the income distribution.

⁸Since second-degree inverse stochastic dominance is equivalent to second-degree stochastic dominance, we will simply refer to this criterion as second-degree dominance.

⁹See Weymark (1981) for an alternative axiomatic justification of W defined by (2.1), when F is considered to be a discrete distribution function.

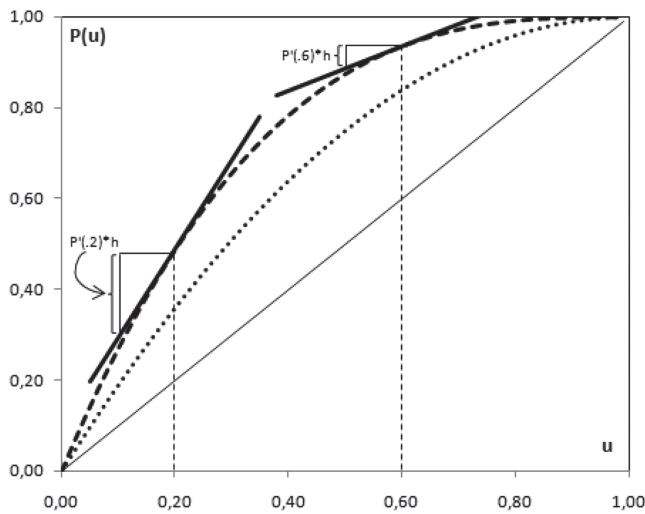


FIGURE 1 Examples of the preference function $P(\cdot)$ that preserves third (dotted) and fourth degree (dashed) upward inverse stochastic dominance. Note: the weight assigned to individuals at rank u equals the derivative of P at u

This example is chosen because incomes are observable and because they are interpersonally comparable, as opposed to individuals' utilities, and thus typically the argument of inequality measures such as the Gini coefficient.¹⁰

2.1.3 | Relation between second-degree dominance and rank-dependent welfare functions

As demonstrated by Yaari (1988), the social welfare functions W_P are consistent with the condition of second-degree stochastic dominance if and only if $P'(t) > 0$ and $P''(t) < 0$. Throughout the paper, we will only consider social preferences that are consistent with second-degree stochastic dominance. Additionally, it can be useful to impose a set of boundary conditions, restricting the social planner's preference function to be given by

$$\mathcal{P} = \{P : P'(t) > 0 \text{ and } P''(t) < 0 \text{ for all } t \in (0, 1), P'(1) = P(0) = 0, P(1) = 1\}.$$

These boundary conditions ensure that $0 \leq W_P \leq \mu_F$ and that $W_P = \mu_F$ if and only if F is the egalitarian distribution. Thus, W_P can be interpreted as the equally distributed equivalent income (EDEI).¹¹ If we compare distributions with equal means, the condition of second-degree stochastic dominance is identical to the Pigou–Dalton transfer principle, which states that an income transfer from a richer to a poorer individual reduces income inequality, provided that their ranks in the income distribution are unchanged.

The general family of social welfare functions W_P represents a preference relation defined on the set of distribution functions. The preference function P assigns weights to the incomes of the individuals in accordance with their rank in the income distribution. Therefore, the functional form of P reveals the attitude toward inequality of a social planner who employs W_P to judge between distribution functions. An inequality neutral planner would choose $P(t) = t$, which means that $W_P(F) = \mu_F$. Figure 1 draws two examples of P and marks the associated weights at ranks $u = 0.2$ and $u = 0.6$. The weight assigned to individuals at rank u equals the derivative of P at u . Note that the preference function must be concave and lie above the diagonal to ensure that W_P satisfies second-degree dominance.

2.1.4 | Interpretation

A normative interpretation of the social welfare function in (2.1) can be made in a theory for ranking distribution functions, as above, or as a value judgement of the trade-off between the mean and (in)equality in the distributions. By defining the ordering relation \succsim on the set of Lorenz curves rather than on the set of distribution functions, Aaberge (2001)

¹⁰Of course, the choice of argument for the social welfare function matters for its economic interpretation and normative justification. For example, while W_P is homogeneous of degree 1 in its argument (utilities or incomes), there is nothing in the rank-dependent approach that rules out using utilities as the argument, such that the social welfare function becomes $W_P(F) = \int_0^1 P'(t)u [F^{-1}(t)] dt$. With ordinary concave utility functions, W_P would then, of course, no longer be homogeneous of degree 1 in income.

¹¹See Atkinson (1970) for a discussion of the EDEI.

demonstrated that \succsim can be represented by the following family of rank-dependent measures of inequality:

$$J_P(F) = 1 - \frac{1}{\mu_F} \int_0^1 P'(u)F^{-1}(u)du, \tag{2.2}$$

where $P'(u)$ is the derivative of a preference function that is a member of the set \mathcal{P} introduced above. Since $J_P(F) = 1 - P'(1)$ when all incomes are received by one individual (complete inequality) and $J_P(F) = 1 - P(1) + P(0)$ when everyone receives the same income (complete equality), the boundary conditions of \mathcal{P} ensures that J_P takes values in the unit interval $[0, 1]$.

Following Ebert (1987), the social welfare function defined by (2.1) can then be expressed as

$$W_P(F) = \mu_F(1 - J_P(F)). \tag{2.3}$$

Equation (2.1) defines W_P as a weighted average of individual incomes where the weights decrease with the individual's rank in the income distribution. Equation (2.3) shows directly how W_P reflects the trade-off between the mean and (in)equality in the distribution of income. The product $\mu_F J_P(F)$ is a measure of the loss in social welfare due to inequality in the distribution of income.

2.1.5 | Parametric subfamilies

In addition to the qualitative ranking of distributions, it is often helpful to quantify the difference between distributions in terms of social welfare. This may, for instance, help evaluate whether the estimated effects of a costly reform are sufficiently large to make up for the costs.

To quantify social welfare, it is necessary to work with parametric social welfare functions. The best known member of W_P is obtained by inserting for $P(t) = 2t - t^2$ in (2.2) and (2.3), in which case $J_P(F)$ is equal to the Gini coefficient and $W_P(F)$ is equal to the much used Gini social welfare function (see Sen, 1974). More generally, by choosing a parametric specification of P , we can derive alternative parametric subfamilies of W_P .

If the preference function is defined by

$$P_{1k}(t) = 1 - (1 - t)^{k-1}, \quad k > 2, \tag{2.4}$$

then J_P becomes equal to the extended Gini family of inequality measures (Donaldson & Weymark, 1980) defined by

$$\begin{aligned} G_k(F) &= 1 - \frac{k-1}{\mu_F} \int_0^1 (1-t)^{k-2} F^{-1}(t) dt \\ &= \frac{1}{\mu_F} \int_0^\infty [1 - F(y)] [1 - (1 - F(y))^{k-2}] dx, \quad k > 2, \end{aligned} \tag{2.5}$$

where $G_3(F)$ is the Gini coefficient.¹² Inserting (2.5) in (2.3), W_P becomes equal to the extended Gini family of social welfare functions, defined by

$$W_{G_k}(F) = \int_0^\infty (1 - F(y))^{k-1} dy = \mu_F [1 - G_k(F)], \quad k > 2. \tag{2.6}$$

If the preference function is instead defined by

$$P_{2k}(t) = \frac{(k-1)t - t^{k-1}}{k-2}, \quad k > 2, \tag{2.7}$$

¹²See Aaberge (2001) for an axiomatic justification for this family of inequality measures.

then J_P becomes equal to the Lorenz family of inequality measures (Aaberge, 2000), defined by

$$\begin{aligned} D_k(F) &= 1 - \frac{k-1}{\mu_F(k-2)} \int_0^1 (1-t^{k-2})F^{-1}(t)dt \\ &= \frac{1}{\mu_F(k-2)} \int_0^\infty F(x)(1-F^{k-2}(x))dx, \quad k > 2, \end{aligned} \quad (2.8)$$

where $D_3(F)$ is the Gini coefficient. Inserting (2.8) for $J_P(F)$ in (2.3), W_P becomes equal to the Lorenz family of social welfare functions

$$W_{D_k}(F) = \frac{k-1}{k-2}\mu_F - \frac{1}{k-2} \int_0^\infty (1-F^{k-1}(x))dx = \mu_F [1 - D_k(F)], \quad k > 2. \quad (2.9)$$

Aaberge (2000) proved that $\{G_k(F) : k = 3, 4, \dots\}$ and $\{D_k(F) : k = 3, 4, \dots\}$ uniquely determine the Lorenz curve L . Since there is one-to-one correspondence between the distribution function F and the associated pair $\{\mu_F, L\}$, it follows that $\{\mu_F, W_{G_k}(F) : k = 3, 4, \dots\}$ and $\{\mu_F, W_{D_k}(F) : k = 3, 4, \dots\}$ uniquely determine the distribution function F , which means that no information is lost by working directly with either of these parametric subfamilies and the mean.

2.2 | Third-degree dominance and social welfare

When distribution functions intersect and second-degree dominance does not provide an unambiguous ranking of distribution functions, weaker criteria are required. This subsection considers third-degree inverse stochastic dominance and characterizes its relationship to W_P . We consider first the criterion of third-degree upward dominance, after which we introduce and analyze the criterion of third-degree downward dominance.

2.2.1 | Upward dominance and social welfare

Let the function associated with second-degree inverse stochastic dominance be defined by

$$\Lambda_F^2(u) = \int_0^u F^{-1}(t)dt, \quad u \in [0, 1], \quad (2.10)$$

where the superscript 2 refers to inverse stochastic dominance of second degree. To define third-degree upward inverse stochastic dominance, we use the notation

$$\Lambda_F^3(u) = \int_0^u \Lambda_F^2(t)dt = \int_0^u F^{-1}(t) \int_t^u dsdt = \int_0^u (u-t)F^{-1}(t)dt, \quad u \in [0, 1], \quad (2.11)$$

which follows by inserting (2.10) in (2.11) and interchanging the order of integration in the third term.

Definition 2.2. A distribution F_1 is said to third-degree upward inverse stochastic dominate a distribution F_0 if and only if $\Lambda_{F_1}^3(u) \geq \Lambda_{F_0}^3(u)$ for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

From Equation (2.11), it is clear that the criterion of third-degree upward inverse stochastic dominance compares weighted sums of incomes, where the weights decrease linearly with the rank in the income distribution.

2.2.2 | Interpretation

Equation (2.3) shows how the rank-dependent social welfare function W_P can be interpreted as reflecting the trade-off between the mean and (in)equality in the distribution of income. Zoli (1999, Lemma 2) noted that third-degree upward inverse stochastic dominance has a similar interpretation, and we will show that all inverse stochastic dominance relations can be given analogous interpretations. For completeness and to establish notation, we will first reproduce Zoli's result for third-degree upward inverse stochastic dominance.

Let H be the conditional distribution function defined by $H(y; u) = Pr(Y \leq y | Y \leq F^{-1}(u)) = F(y)/u$, for any $y \leq F^{-1}(u)$. The quantile-specific lower tail mean is defined by

$$\mu_F(u) = \mu_H = \int_0^{F^{-1}(u)} y dH(y; u) = \frac{\int_0^u F^{-1}(t) dt}{u}, \tag{2.12}$$

and the quantile-specific lower tail Gini coefficient is defined by

$$G_3(u; F) = \frac{1}{\mu_H} \int_0^1 (2t - 1) H^{-1}(t; u) dt = \frac{1}{u^2 \mu_F(u)} \int_0^u (2t - u) F^{-1}(t) dt, \tag{2.13}$$

where $H^{-1}(t; u)$ is the left inverse of $H(y; u)$. The quantile-specific lower tail Gini social welfare function is then given by $\mu_F(u) (1 - G_3(u; F))$. This function provides a Gini-based measure of the EDEI in the conditional distribution $H(y; u)$, where $\mu_F(u) G_3(u; F)$ is a measure of the loss in social welfare due to inequality in $H(y; u)$.

The following proposition shows that the criterion of third-degree upward inverse stochastic dominance is equivalent to employing the Gini social welfare function to compare the welfare of individuals located in the lower tail of each quantile in the distributions. That is, a distribution F_1 is said to third-degree upward inverse stochastic dominate F_0 if and only if the Gini-based measure of the EDEI among the poorest u percent is higher in F_1 as compared with F_0 for all u . This sequential comparison of the quantile-specific social welfare functions tells us not only whether F_1 third-degree downward dominates F_0 according to social welfare but also where in the distributions the differences in social welfare arise. On top of this, we can learn whether the social welfare is larger in the dominating distribution because it has less inequality or because it has a higher mean.

Proposition 2.1 (Zoli, 1999). *Let F_1 and F_0 be members of \mathcal{F} . Then the following statements are equivalent:*

- (i) F_1 third-degree upward inverse stochastic dominates F_0
- (ii) $\mu_{F_1}(u) (1 - G_3(u; F_1)) \geq \mu_{F_0}(u) (1 - G_3(u; F_0))$ for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

Proof. This result follows by noting that

$$\Lambda_F^3(u) = \frac{u^2}{2} \mu_F(u) (1 - G_3(u; F)), \tag{2.14}$$

which is obtained by inserting (2.12) and (2.13) in (2.11). □

2.2.3 | Transfer principle

To provide a normative justification for dominance criteria of third degree, more powerful principles than the Pigou–Dalton transfer principle are needed. To this end, Kolm (1976) introduced the principle of diminishing transfers, which for a fixed difference in income considers a transfer from a richer to a poorer person to be more equalizing the further down in the income distribution it occurs. As indicated by Shorrocks and Foster (1987) and Muliere and Scarsini (1989), the principle of diminishing transfers is, however, not consistent with third-degree upward inverse stochastic dominance. We will instead use an alternative version of the principle of diminishing transfers introduced by Mehran (1976)—and called the principle of positional transfer sensitivity by Zoli (1999)—to characterize third-degree upward inverse stochastic dominance.

To provide a formal definition of the principle of positional transfer sensitivity, it will be useful to introduce the notation $\Delta_s W_P(\delta, h)$, which denotes the change in W_P of a fixed progressive transfer δ from an individual with rank $s + h$ to an individual with rank s . Further, let

$$\Delta_{st}^1 W_P(\delta, h) \equiv \Delta_s W_P(\delta, h) - \Delta_t W_P(\delta, h).$$

We can then define the principle of first-degree DPTS.

Definition 2.3. W_P satisfies the principle of first-degree DPTS if and only if $\Delta_{st}^1 W_P(\delta, h) > 0$, for all $s < t$.

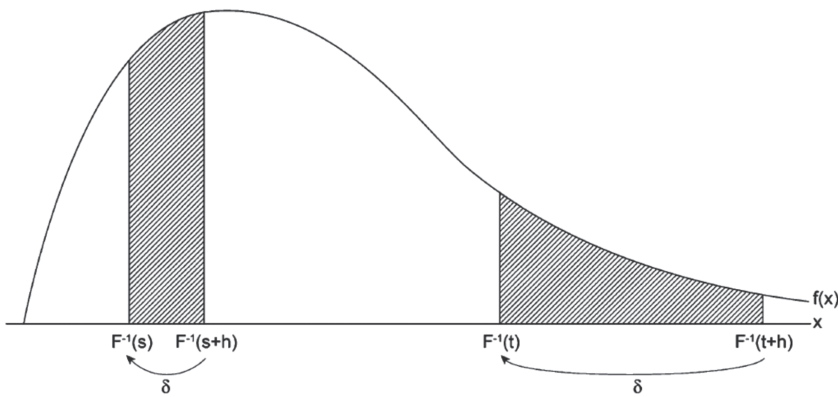


FIGURE 2 Income transfers and the principles of positional transfer sensitivity. *Note:* this figure draws the probability density of a right-skewed income distribution, denoted $f(x)$. We have also drawn two alternative transfers from richer to poorer, one from an individual at rank $t + h$ to an individual at rank t , and another from rank $s + h$ to rank s ; the equal difference in rank h is reflected in the equal size of the shaded areas

To better understand first-degree DPTS and how it relates to the Pigou–Dalton transfer principle, consider Figure 2 where we draw the probability density of a right-skewed income distribution, denoted $f(x)$. We have also drawn two alternative transfers from richer to poorer, one from an individual at rank $t + h$ to an individual at rank t , and another from rank $s + h$ to rank s ; the equal difference in rank h is reflected in the equal size of the shaded areas. Consider first the two transfers in isolation. According to the Pigou–Dalton transfer principle, both transfers should decrease inequality and hence increase welfare. According to first-degree DPTS, given that a fixed transfer takes place between two people with equal difference in ranks, the transfer at lower ranks has a stronger equalizing effect—and thus increases social welfare more—than the transfer at higher ranks. An inequality averse social planner who supports the principle of first-degree DPTS is said to exhibit downside positional inequality aversion of first degree.

2.2.4 | Equivalence result

Let \mathcal{P}_3 be the family of preference functions defined by

$$\mathcal{P}_3 = \{P \in \mathcal{P} : P'''(t) > 0, \text{ for all } t \in (0, 1) \text{ and } P''(1) \leq 0\}. \quad (2.15)$$

The following result provides a characterization of the relationship between third-degree upward inverse stochastic dominance and the general family of welfare functions.¹³

Theorem 2.1. *Let F_1 and F_0 be members of \mathcal{F} . Then the following statements are equivalent:*

- (i) F_1 third-degree upward inverse stochastic dominates F_0 .
- (ii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathcal{P}_3$.
- (iii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathcal{P}$ where W_P satisfies first-degree DPTS.

Proof. See the Supporting Information. □

The equivalence between (i) and (ii) in Theorem 2.1 reveals the least-restrictive set of social welfare functions that allows an unambiguous ranking of distribution functions in accordance with third-degree upward inverse stochastic dominance. This is ensured by imposing the requirement of a positive third-derivative on the preference function P . Further, the equivalence with (iii) provides a normative justification for ranking distribution functions according to third-degree upward dominance.

Note that Theorem 2.1 extends on the previous literature in several ways. Mehran (1976) conjectures that J_P defined by (2.2) satisfies first-degree DPTS if and only if $P'''(t) > 0$, but without providing a formal proof. This result is restated

¹³Note that the equivalence results of Theorem 2.2 do not require any restrictions on the distribution functions and is thus more general than the equivalence result for third-degree stochastic dominance proposed by Whitmore (1970). Muliere and Scarsini (1989) used the i th-degree (upward) inverse stochastic dominance criterion to characterize a particular subfamily of the dual family of social welfare functions.

in the equivalence of (ii) and (iii) in Theorem 2.1 and proven in the Supporting Information. Zoli (1999) shows a similar equivalence as the one between (i) and (ii), but relies on restrictions about the distribution functions being compared. Specifically, he characterizes the relationship between rank-dependent social welfare functions and third-degree inverse stochastic dominance of F_1 over F_2 in situations in which $\mu_{F_1} \geq \mu_{F_2}$. By comparison, we impose no restrictions on the distribution functions being compared. Additionally, our results differ from that of Zoli in that we impose boundary conditions on the set \mathcal{P} ; these conditions ensure that the measures of inequality take values in the unit interval $[0, 1]$, and that the social welfare measures can be interpreted as the EDEI.

2.2.5 | Downward dominance and social welfare

Section 2.2.1 demonstrated that a social planner who supports the criterion of third-degree upward inverse stochastic dominance exhibits aversion to downside inequality. In some cases, however, the researcher may want ranking criteria that are more sensitive to income differences in the upper part of the distribution.

To focus attention on differences in the upper part of the distribution, we now consider an alternative transfer principle and show how it corresponds to the criterion of third-degree *downward* inverse stochastic dominance. This criterion is obtained by aggregating the integrated inverse distribution function from above, rather than from below as in upward dominance. To define third-degree downward dominance, we use the notation

$$\tilde{\Lambda}_F^3(u) = \int_u^1 \Lambda_F^2(t)dt = (1 - u)\mu - \int_u^1 (t - u)F^{-1}(t)dt, \quad u \in [0, 1], \tag{2.16}$$

where the second equality follows from inserting (2.10) for Λ_F^2 and by interchanging the order of integration.

Definition 2.4. A distribution F_1 is said to third-degree downward inverse stochastic dominate a distribution F_0 if and only if $\tilde{\Lambda}_{F_1}^3(u) \geq \tilde{\Lambda}_{F_0}^3(u)$, for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

From Equation (2.16), it is clear that the criterion of third-degree downward dominance compares the weighted sums of incomes, where the weights decrease linearly with the rank in the income distribution.

2.2.6 | Interpretation

Equation (2.3) shows how W_P can be interpreted as reflecting the trade-off between the mean and (in)equality in the distribution of income. We now show that third-degree downward dominance has an analogous interpretation.

Let \tilde{H} be the conditional distribution function defined by $\tilde{H}(y; u) = Pr(Y \leq y | Y \geq F^{-1}(u)) = (F(y) - u)/(1 - u)$, for any $y \geq F^{-1}(u)$. The quantile-specific upper tail mean is defined by

$$\tilde{\mu}_F(u) = \mu_{\tilde{H}} = \int_{F^{-1}(u)}^1 y d\tilde{H}(y; u) = \frac{\int_u^1 F^{-1}(t)dt}{1 - u}, \tag{2.17}$$

and the quantile-specific upper tail Gini coefficient is defined by

$$D_3(u; F) = \frac{1}{\mu_{\tilde{H}}} \int_0^1 (2t - 1)\tilde{H}^{-1}(t; u)dt = \frac{\int_u^1 (2t - u - 1)F^{-1}(t)dt}{(1 - u)^2 \tilde{\mu}_F(u)}, \tag{2.18}$$

where $\tilde{H}^{-1}(t; u)$ is the left inverse of $\tilde{H}(y; u)$. The quantile-specific upper tail Gini social welfare function is then given by $\tilde{\mu}_F(u) (1 - D_3(u; F))$, which can be interpreted as the Gini-based measure of the EDEI for the conditional distribution $\tilde{H}(y; u)$, where $\mu_F(u)D_3(u; F)$ is a measure of the loss in social welfare due to inequality in $\tilde{H}(y; u)$.

The following proposition shows that the criterion of third-degree downward dominance is a sequential comparison of a weighted sum of the mean income of the poorest u percent, and the Gini social welfare of the richest $(1 - u)$ percent of the population.

Proposition 2.2. Let F_1 and F_0 be members of \mathcal{F} . Then the following statements are equivalent:

- (i) F_1 third-degree downward inverse stochastic dominates F_0 .
- (ii) $u\mu_{F_1}(u) + \frac{(1-u)}{2}\tilde{\mu}_{F_1}(u)(1 - D_3(u; F_1)) \geq u\mu_{F_0}(u) + \frac{(1-u)}{2}\tilde{\mu}_{F_0}(u)(1 - D_3(u; F_0))$ for all $u \in [0, 1)$ and the inequality holds strictly for some $u \in (0, 1)$.

Proof. This result is obtained by noting that $\mu_F = u\mu_F(u) + (1 - u)\tilde{\mu}_F(u)$ and

$$\tilde{\Lambda}_F^3(u) = u(1 - u)\mu_F(u) + \frac{(1 - u)^2}{2}\tilde{\mu}_F(u)(1 - D_3(u; F)), \quad (2.19)$$

which follows by inserting (2.17) and (2.18) in (2.16). \square

2.2.7 | Transfer principle

To provide a normative justification for downward dominance of third degree, more powerful principles than the Pigou–Dalton transfer principle are needed. We will employ the principle of *upside* positional transfer sensitivity—introduced by Aaberge (2009) for analyzing Lorenz dominance—to characterize third-degree downward inverse stochastic dominance.

As above, let $\Delta_s W_P(\delta, h)$ denote the change in W_P of a fixed progressive transfer δ from an individual with rank $s + h$ to an individual with rank s , and let

$$\Delta_{st}^1 W_P(\delta, h) \equiv \Delta_s W_P(\delta, h) - \Delta_t W_P(\delta, h).$$

We can then define the principle of first-degree UPTS.

Definition 2.5. W_P satisfies the principle of first-degree UPTS if and only if $\Delta_{st}^1 W_P(\delta, h) < 0$, for all $s < t$.

To better understand first-degree UPTS and how it relates to the Pigou–Dalton transfer principle and first-degree DPTS, revisit Figure 2. We have drawn two alternative transfers from richer to poorer: one from an individual at rank $t + h$ to an individual at rank t , and another from rank $s + h$ to rank s ; the equal difference in rank h is reflected in the equal size of the shaded areas. This implies that the number of people between the donor and the receiver is the same.

Consider first the two transfers in isolation. According to the Pigou–Dalton transfer principle, both transfers should decrease inequality and hence increase welfare. According to first-degree UPTS, given that a fixed transfer takes place between two persons with equal difference in ranks, the transfer at lower ranks has a weaker equalizing effect—and thus increases social welfare less—than the transfer at higher ranks. An inequality averse social planner that supports the principle of first-degree UPTS is therefore said to exhibit upside positional inequality aversion of first degree. The choice between DPTS and UPTS clarifies, therefore, whether equalizing transfers in the lower part of the distribution should be considered more or less important for social welfare as compared with equalizing transfers in the upper part of the distribution.

2.2.8 | Equivalence result

Let \mathcal{P}_3 be the family of preference functions defined by

$$\tilde{\mathcal{P}}_3 = \{P \in \mathcal{P} : P'''(t) < 0 \text{ for all } t \in (0, 1) \text{ and } P''(0) \leq 0\}. \quad (2.20)$$

The following result provides a characterization of the relationship between third-degree downward inverse stochastic dominance and the general family of welfare functions.

Theorem 2.2. Let F_1 and F_0 be members of \mathcal{F} . Then the following statements are equivalent,

- (i) F_1 third-degree downward inverse stochastic dominates F_0 .
- (ii) $W_P(F_1) > W_P(F_0)$ for all $P \in \tilde{\mathcal{P}}_3$.
- (iii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathcal{P}$ where W_P satisfies first-degree UPTS.

Proof. See the Supporting Information. □

The equivalence between (i) and (ii) in Theorem 2.2 reveals the least-restrictive set of social welfare functions that allows an unambiguous ranking of distribution functions in accordance with third-degree downward inverse stochastic dominance. This is ensured by imposing the requirement of a negative third-derivative on the preference function P . Further, the equivalence with (iii) provides a normative justification for ranking distribution functions according to third-degree downward dominance. By comparing (iii) in Theorems 2.1 and 2.2, it is clear that the choice between third-degree upward dominance and third-degree downward dominance depends on whether income differences between poorer individuals are viewed as more or less important for social welfare as compared with income differences between richer individuals.

2.3 | Dominance of i th-degree and social welfare

In some cases, neither upward nor downward dominance of third degree allows an unambiguous ranking of the distribution functions under comparison. This subsection therefore introduces the full hierarchical sequences of nested inverse stochastic dominance criteria, allowing ranking of a given set of distribution functions. We further characterize the relationship between W_p and upward or downward dominance of any degree.

To define upward inverse stochastic dominance of degree i , we use the notation

$$\begin{aligned} \Lambda_F^i(u) &= \int_0^u \Lambda_F^{i-1}(t)dt = \frac{1}{(i-3)!} \int_0^u (u-t)^{i-3} \Lambda_F^2(t)dt \\ &= \frac{1}{(i-2)!} \int_0^u (u-t)^{i-2} F^{-1}(t)dt, \quad i = 3, 4, \dots \end{aligned} \tag{2.21}$$

To define downward inverse stochastic dominance of degree i , we use the notation

$$\begin{aligned} \tilde{\Lambda}_F^i(u) &= \int_u^1 \tilde{\Lambda}_F^{i-1}(t)dt = \frac{1}{(i-3)!} \int_u^1 (t-u)^{i-3} \Lambda_F^2(t)dt \\ &= \frac{1}{(i-2)!} \left[(1-u)^{i-2} \mu_F - \int_u^1 (t-u)^{i-2} F^{-1}(t)dt \right] \quad i = 3, 4, \dots \end{aligned} \tag{2.22}$$

Definition 2.6. A distribution F_1 is said to i th-degree upward inverse stochastic dominate F_0 if and only if $\Lambda_{F_1}^i(u) \geq \Lambda_{F_0}^i(u)$, for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

Definition 2.7. A distribution F_1 is said to i th-degree downward inverse stochastic dominate F_0 if and only if $\tilde{\Lambda}_{F_1}^i(u) \geq \tilde{\Lambda}_{F_0}^i(u)$, for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

From Equations (2.21) and (2.22), it is clear that the criteria of both i th-degree upward and downward dominance compare the weighted sums of incomes, where the weights decrease with the rank in the income distribution.¹⁴ As will be demonstrated below, however, the choice between higher degree of upward and downward dominance clarifies whether preferences of the social planner gives priority to reduction of inequality in the lower or the upper part of the income distribution.

2.3.1 | Interpretation

We now show that upward and downward dominance of degree i can be interpreted as reflecting trade-offs between the mean and (in)equality in the distribution of income. To this end, we employ the two parametric subfamilies of W_p

¹⁴Note that Definition 2.6 does not require any restrictions on the distribution functions and is therefore less restrictive than the analogous definition of i th-degree stochastic dominance proposed by Chew (1983).

presented above: The first is the extended Gini family of social welfare functions $W_{G_k}(F)$, defined by Equation (2.6); the second is the Lorenz family of social welfare functions $W_{D_k}(F)$, defined by Equation (2.9).

The quantile-specific lower tail extended Gini family of inequality measures is defined by

$$G_i(u; F) = 1 - \frac{i-1}{\mu_H} \int_0^1 (1-t)^{i-2} H^{-1}(t) dt = 1 - \frac{i-1}{u^{i-1} \mu_F(u)} \int_0^u (u-t)^{i-2} F^{-1}(t) dt, \tag{2.23}$$

and the associated quantile-specific lower tail extended Gini family of social welfare functions can then be expressed as $\mu_F(u)(1 - G_i(u; F))$, where $\mu_F(u)G_i(u; F)$ is a measure of the loss in social welfare due to inequality in $H(y; u)$.

The quantile-specific upper tail Lorenz family of inequality measures is defined by

$$\begin{aligned} D_i(u; F) &= 1 - \frac{i-1}{(i-2)\mu_{\tilde{H}}} \int_0^1 (1-t^{i-2}) \tilde{H}^{-1}(t) dt \\ &= 1 - \frac{i-1}{(i-2)(1-u)^{i-1} \tilde{\mu}_F(u)} \int_u^1 [(1-u)^{i-2} - (t-u)^{i-2}] F^{-1}(t) dt, \end{aligned} \tag{2.24}$$

and the associated quantile-specific upper tail Lorenz family of social welfare functions can then be expressed as $\tilde{\mu}_F(u)(1 - D_i(u; F))$, where $\mu_F(u)D_i(u; F)$ is a measure of the loss in social welfare due to inequality in $\tilde{H}(y; u)$.

Proposition 2.3 shows that the criterion of i th-degree upward dominance is equivalent to employing the Gini social welfare function of order i to compare welfare among individuals located at the lower tail of each quantile of the distributions. Proposition 2.4 shows that the criterion of third-degree downward dominance corresponds to a sequential comparison of a weighted sum of the mean income of the poorest u percent, and the social welfare of the richest $(1 - u)$ percent of the population according to the Lorenz social welfare function of order i .

Proposition 2.3. *Let F_0 and F_1 be members of \mathcal{F} . Then the following statements are equivalent:*

- (i) F_1 i th-degree upward inverse stochastic dominates F_0 .
- (ii) $\mu_{F_1}(u)(1 - G_i(u; F_1)) \geq \mu_{F_0}(u)(1 - G_i(u; F_0))$ for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

Proof. This result is obtained by noting that

$$\Lambda_F^i(u) = \frac{u^{i-1}}{(i-1)!} \mu_F(u)(1 - G_i(u; F)), \tag{2.25}$$

which follows by inserting (2.12) and (2.23) in (2.21). □

Proposition 2.4. *Let F_0 and F_1 be members of \mathcal{F} . Then the following statements are equivalent:*

- (i) F_1 i th-degree downward inverse stochastic dominates F_0 .
- (ii) $u\mu_{F_1}(u) + \frac{(i-2)}{(i-1)}(1-u)\tilde{\mu}_{F_1}(u)(1 - D_i(u; F_1)) \geq u\mu_{F_0}(u) + \frac{(i-2)}{(i-1)}(1-u)\tilde{\mu}_{F_0}(u)(1 - D_i(u; F_0))$ for all $u \in [0, 1]$, and the inequality holds strictly for some $u \in (0, 1)$.

Proof. This result is obtained by noting that

$$\tilde{\Lambda}_F^i(u) = \frac{u(1-u)^{i-2}}{(i-2)!} \mu_F(u) + \frac{(i-2)(1-u)^{i-1}}{(i-1)!} \tilde{\mu}_F(u)(1 - D_i(u; F)), \tag{2.26}$$

which follows by inserting (2.17) and (2.24) in (2.22). □

2.3.2 | Transfer principles

To provide a normative justification for upward (downward) dominance of degree i , we employ generalizations of the principle of downside (upside) positional transfer sensitivity. As above, let $\Delta_s W_P(\delta, h)$ denote the change in W_P of a fixed progressive transfer δ from an individual with rank $s + h$ to an individual with rank s , and let $\Delta_{st}^1 W_P(\delta, h) = \Delta_s W_P(\delta, h) - \Delta_t W_P(\delta, h)$. Further, let

$$\Delta_{st}^i W_P(\delta, h_1, h_2, \dots, h_i) \equiv \Delta_{st}^{i-1} W_P(\delta, h_1, h_2, \dots, h_{i-1}) - \Delta_{s+h_i, t+h_i}^{i-1} W_P(\delta, h_1, h_2, \dots, h_{i-1}), \tag{2.27}$$

for $i = 2, 3, \dots$, denote the difference in the change in social welfare from a series of progressive transfers at lower ranks (s) compared with higher ranks (t) in the income distribution. We can then define the principles of DPTS and UPTS of degree i .

Definition 2.8. W_P satisfies the principle of DPTS of degree i if and only if, for all $k = 1, 2, \dots, i$

$$(-1)^k \Delta_{st}^k W_P(\delta, h) > 0, \text{ when } s < t.$$

Definition 2.9. W_P satisfies the principle of UPTS of degree i if and only if, for all $k = 1, 2, \dots, i$

$$\Delta_{st}^k W_P(\delta, h) > 0, \text{ when } s < t.$$

Given two alternative sequences of fixed transfers between people with equal difference in ranks, i th degree UPTS (DPTS) states that the sequence of transfers at lower ranks have a stronger (weaker) equalizing effect—and thus increase social welfare more (less)—than the sequence of transfers at higher ranks. Further, a social planner that supports the principle of i th degree UPTS (DPTS) exhibits relatively higher inequality aversion in the lower (upper) parts of the distribution, as compared with a social planner that supports the principle of $(i - 1)$ th-degree UPTS (DPTS). An inequality averse social planner that supports the principle of i th-degree UPTS (DPTS) is therefore said to exhibit downside (upside) positional inequality aversion of degree i .¹⁵ Because UPTS (DPTS) of degree i are stronger criteria than UPTS (DPTS) of degree $i - 1$, it seems natural that a social planner who supports the latter will also support the former.

2.3.3 | Equivalence result

Let $P^{(j)}$ be the j th-degree derivative of P . The family of preference functions \mathcal{P}_i is defined by

$$\mathcal{P}_i = \left\{ P \in \mathcal{P} : \begin{aligned} &(-1)^{i-1} P^{(i)}(t) > 0 \\ &\text{and } (-1)^{j-1} P^{(j)}(1) \geq 0 \text{ for all } j = 2, 3, \dots, i - 1 \end{aligned} \right\}, \tag{2.28}$$

whereas the family of preference functions $\tilde{\mathcal{P}}_i$ is defined by

$$\tilde{\mathcal{P}}_i = \left\{ P \in \mathcal{P} : \begin{aligned} &P^{(i)}(t) < 0 \\ &\text{and } P^{(j)}(0) \leq 0 \text{ for all } j = 2, 3, \dots, i - 1 \end{aligned} \right\}. \tag{2.29}$$

The following theorems provide a characterization of the relationship between i th-degree upward and downward inverse stochastic dominance and the general family of welfare functions.

¹⁵Note that i th-degree DPTS can be considered as an alternative to the i th-degree transfer principle introduced by Fishburn and Willig (1984) as an extension of Kolm's principle of diminishing transfers.

Theorem 2.3. Let F_1 and F_0 be members of \mathcal{F} . Then for $i = 3, 4, \dots$, the following statements are equivalent:

- (i) F_1 i th-degree upward inverse stochastic dominates F_0 .
- (ii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathcal{P}_i$.
- (iii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathcal{P}$ where W_P satisfies DPTS of degree $i - 2$.

Proof. See the Supporting Information. □

Theorem 2.4. Let F_1 and F_0 be members of \mathcal{F} . Then for $i = 3, 4, \dots$, the following statements are equivalent

- (i) F_1 i th-degree downward inverse stochastic dominates F_0 .
- (ii) $W_P(F_1) > W_P(F_0)$ for all $P \in \tilde{\mathcal{P}}_i$.
- (iii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathcal{P}$ where W_P satisfies UPTS of degree $i - 2$.

Proof. See the Supporting Information. □

Remark. The dominance relations are transitive. To see this, assume

- (i) F_1 i th-degree upward (downward) dominates F_2 .
- (ii) F_2 $(i - k)$ th degree upward (downward) dominates F_3 , for $k \in \{0, 1, \dots, i - 1\}$.

From Equations (2.21) and (2.22), it follows that $\Lambda_{F_1}^{i-1}(u) \geq \Lambda_{F_2}^{i-1}(u)$ for all u implies $\Lambda_{F_1}^i(u) \geq \Lambda_{F_2}^i(u)$ for all u . Conditions (i) and (ii) therefore imply that F_1 i th-degree upward (downward) inverse stochastic dominates F_3 .

The equivalence between (i) and (ii) in Theorems 2.3 and 2.4 reveals the least-restrictive set of social welfare functions that allows an unambiguous ranking of distribution functions in accordance with i th degree upward or downward inverse stochastic dominance.

Upward dominance of degree i is ensured by imposing positive (negative) i th-degree derivative if i is odd (even) on the preference function P . Together with the boundary condition, this makes sure that the implied set of weights becomes more progressive as i increases. This means that a social planner who employs the criterion of i th-degree upward dominance pays more attention to inequality in the lower than in the upper part of the income distribution as compared with a social planner who employs the criterion of $(i - 1)$ th-degree upward dominance. By comparison, downward dominance of degree i is ensured by imposing negative i th-degree derivative on the preference function P . Together with the boundary condition, this makes sure that the implied set of weights becomes more progressive as i increases. This means that a social planner who employs the criterion of i th-degree downward dominance pays more attention to inequality in the upper than in the lower part of the income distribution as compared with a social planner who employs the criterion of $(i - 1)$ th-degree downward dominance.

The equivalence between (i) and (iii) in Theorems 2.3 and 2.4 provides normative justification for ranking distribution functions according to i th-degree upward and downward dominance. By comparing (iii) in these two theorems, it is clear that the choice between i th-degree upward dominance and i th-degree downward dominance depends on whether income differences between poorer individuals are viewed as more or less important for social welfare as compared with income differences between richer individuals.

The results in Theorems 2.2 and 2.4 are related to but distinct from Theorems 3.1A and 3.2B of Aaberge (2009), both in terms of implications and of proofs. The latter theorems state that i th-degree Lorenz-dominance (upward or downward) of distribution F_1 over F_2 is equivalent to $J_P(F_1) < J_P(F_2)$ for an appropriate refinement of the set of admissible weighting functions P . If $\mu_{F_1} = \mu_{F_2}$, then the social welfare ordering within this set using $W_P(F)$ follows immediately. If $\mu_{F_1} \neq \mu_{F_2}$, however, there is no clear implication. In particular, if $\mu_{F_1} < \mu_{F_2}$, the social welfare ordering could easily be reversed compared with the Lorenz ordering, and if $\mu_{F_1} > \mu_{F_2}$, the refinements on the set P will often be too strict.

2.4 | The limits of the dominance criteria

The proposed sequences of dominance criteria along with Theorems 2.3 and 2.4 suggest two complementary strategies for successively narrowing the general family of social welfare functions in order to unambiguously rank a given set of distribution functions. Though the theorems are only valid for finite i , to understand their normative implications, it is helpful to consider the limits of the sequences of dominance criteria.

As $i \rightarrow \infty$, we get from Equations (2.21) and (2.22)

$$(i - 1)! \Lambda^i(u) \rightarrow \begin{cases} 0, & 0 \leq u < 1, \\ F^{-1}(0+), & u = 1, \end{cases} \tag{2.30}$$

$$(i - 2)! \tilde{\Lambda}^i(u) \rightarrow \begin{cases} \mu_F, & u = 0, \\ 0, & 0 < u \leq 1, \end{cases} \tag{2.31}$$

where $F^{-1}(0+)$ denotes the lowest income in F . In the limit, upward and downward inverse stochastic dominance therefore depend only on the income of the worst-off income recipient and the average income, respectively.

The highest degree of downside inequality aversion is achieved when focus is exclusively turned to the situation of the poorest in the population. In this case, the social welfare function corresponds to the *Rawlsian maximin criterion*. By contrast, the highest degree of upside inequality aversion is achieved when focus is exclusively turned to the mean income. The mean income criterion (indifferent to inequality) is “dual” to the Rawlsian maximin criterion (cares only about the poorest individual) in the sense that it is compatible with the limiting case of downward inverse stochastic dominance. In this case, the distribution function for which the mean income is largest is preferred, regardless of all other differences.

3 | INVERSE STOCHASTIC DOMINANCE AND PARAMETRIC FAMILIES OF SOCIAL WELFARE FUNCTIONS

Until now, the results and discussion have centered on characterizing the relationship between inverse stochastic dominance criteria and W_p in the ranking of intersecting distribution functions. This section extends our approach to not only rank distributions but also quantify the social welfare level of a dominating distribution as compared with a dominated distribution. To this end, we employ the two parametric subfamilies of W_p presented above: the first is the extended Gini family of social welfare functions $W_{G_i}(F)$, defined by Equation (2.6); the second is the extended Lorenz family of social welfare functions $W_{D_i}(F)$, defined by Equation (2.9). Because $\{\mu_F, W_{G_i}(F) : i = 3, 4, \dots\}$ and $\{\mu_F, W_{D_i}(F) : i = 3, 4, \dots\}$ uniquely determine the distribution function F (see Aaberge, 2000), no information is lost by working directly with either of these parametric subfamilies and the mean.

3.1 | Upward dominance and the extended Gini family

Corollary 3.1 sorts the members of the Gini family of social welfare functions into subfamilies according to their relationship to upward inverse stochastic dominance. This allows us to identify the largest subfamily of $W_{G_i}(F)$ that ranks consistently with upward dominance of a given degree and quantify the social welfare level of the dominating distribution as compared with the dominated distribution. From Theorem 2.3, we get the following result.

Corollary 3.1. *Let F_1 and F_0 be members of \mathcal{F} . Then for $i = 2, 3 \dots$,*

- (i) F_1 i th-degree upward inverse stochastic dominates F_0 implies
- (ii) $W_{G_k}(F_1) > W_{G_k}(F_0)$ for $k > i$.

Remark 1. The extended Gini family of social welfare functions has the following properties:

- (i) W_{G_i} obeys the Pigou–Dalton principle of transfers for $i > 2$.
- (ii) W_{G_i} obeys the principles of DPTS up to and including $(i - 2)$ th-degree for $i = 3, 4, \dots$
- (iii) The sequence $\{W_{G_i}\}$ approaches μ_F when $i \rightarrow 2$
- (iv) The sequence $\{W_{G_i}\}$ approaches the Rawlsian maximin criterion when $i \rightarrow \infty$.

The left panel of Figure 3 displays the preference function $P_{1k}(t)$, associated with the Gini family of social welfare functions, defined by (2.4) when $k = 3, k = 4$, and $k = 10$. As we increase the degree of upward dominance preserved by W_{G_k} , we see how the preference function becomes more sensitive to income differences in the lower part of the distribution. This is also illustrated in panel (a) of Table 1. This table shows how different social welfare functions within the Gini family assigns weights to incomes at selected quantiles relative to the weight assigned to the median income, both when

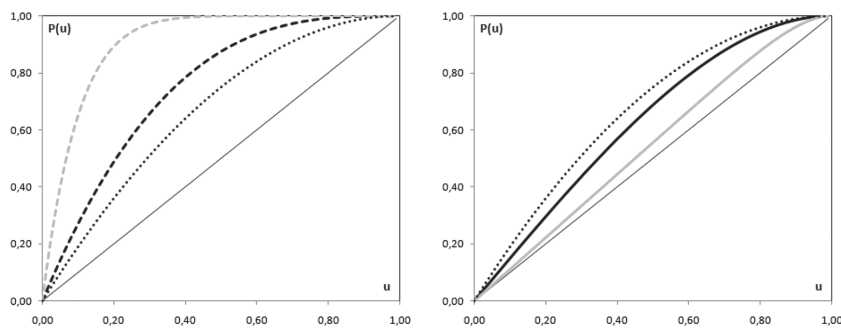


FIGURE 3 Examples of the preference function P that preserves second-, third-, and 10th-degree inverse stochastic dominance, upwards (left panel) and downwards (right panel). Note: the weight assigned to individuals at rank u equals the derivative of P at u . The parametric forms of P are defined in Section 2.1

	Quantile					
	0.01	0.05	0.30	0.70	0.95	0.99

Panel (a): Gini social welfare function (upward)

$k = 2$	1.00	1.00	1.00	1.00	1.00	1.00
$k = 3$	2.00	1.90	1.40	0.60	0.10	0+
$k = 4$	4.00	3.61	1.96	0.36	0.01	0+
$k = 5$	8.00	6.86	2.74	0.22	0.00	0+
$k = 6$	16.00	13.03	3.84	0.13	0.00	0+
$k \rightarrow \infty$	∞	0	0	0	0	0

Panel (b): Lorenz social welfare function (downward)

$k = 3$	2.00	1.90	1.40	0.60	0.10	0+
$k = 4$	1.33	1.33	1.21	0.68	0.13	0+
$k = 5$	1.14	1.14	1.11	0.75	0.16	0+
$k = 6$	1.07	1.07	1.06	0.81	0.20	0+
$k \rightarrow \infty$	1	1	1	1	1	1-

TABLE 1 Weights in W_{G_k} and W_{D_k} at selected quantiles relative to the weight at the median

Note: The weights in the table refer to the derivative P' of the parametric forms of the weighting function P defined in Section 2.1.

$k = 3, 4, 5, 6$ and in the limits as $k \rightarrow 2$ and $k \rightarrow \infty$. The highest degree of downside inequality aversion occurs as $k \rightarrow \infty$, which corresponds to the Rawlsian maximin criterion. At the other extreme, $k \rightarrow 2$ and W_{G_k} equals the mean income.

Consider, for example, the difference between dominance of third and fifth degree. Our propositions imply an equivalence between ranking income distributions according to the extended Gini social welfare function for $k > 3$ and $k > 5$ and upward dominance of third and fifth degree, respectively. Table 1 reveals that the least restrictive parametric welfare function that ranks consistently with third-degree upward dominance, assigns more than 1.9 times as much weight to the poorest 5% and more than 1.4 times as much weight to the poorest 30%, whereas the richest 30% and 5% should receive less than 0.6 and 0.1 times the weight assigned to the median income. By comparison, the least restrictive parametric welfare function that ranks consistently with sixth-degree upward dominance implies quite restrictive social preferences: As compared with the median income, the 5% and 30% poorest must receive more than 16 and 13 times as much weight as the median while the richest 70% and 95% must be assigned less than 0.13 and approximately zero times the weight of the median.

3.2 | Downward dominance and the extended Lorenz family

Corollary 3.2 sorts the members of the Lorenz family of social welfare functions into subfamilies according to their relationship to downward inverse stochastic dominance. This allows us to identify the largest subfamily of $W_{D_i}(F)$ that ranks consistently with downwards dominance of a given degree and quantify the social welfare level of the dominating distribution as compared with the dominated distribution. From Theorem 2.4, we get the following result.

Corollary 3.2. Let F_1 and F_0 be members of \mathcal{F} . Then for $i = 2, 3, \dots$,

- (i) F_1 i th-degree downward inverse stochastic dominates F_0 implies.
- (ii) $W_{D_k}(F_1) > W_{D_k}(F_0)$ for $k > i$.

Remark The extended Lorenz family of social welfare functions has the following properties:

- (i) W_{D_i} obeys the Pigou–Dalton principle of transfers for $i > 2$.
- (ii) W_{D_i} obeys the principles of UPTS up to and including $(i - 2)$ th-degree for $i = 3, 4, \dots$
- (iii) The sequence $\{W_{D_i}\}$ approaches the Bonferroni welfare function $\int [1 - F(x)(1 - \log F(x))] dx$ when $i \rightarrow 2$
- (iv) The sequence $\{W_{D_i}\}$ approaches μ_F as $i \rightarrow \infty$.
- (v) The sequence $\{i(W_{D_i} - \mu_F)\}$ approaches $-F^{-1}(1 -)$ as $i \rightarrow \infty$, which means that the distribution with the lowest maximum income is considered preferable provided that the distributions in question have equal mean income.

The right panel of Figure 3 displays the preference function $P_{2k}(t)$ when $k = 3$, $k = 4$, and $k = 10$. As we increase the degree of downward dominance preserved by W_{D_i} , we see how the preference function becomes more sensitive to income differences in the upper part of the distribution. This is also illustrated in panel (b) of Table 1. This table shows how $P_{2k}(t)$ assigns weights to incomes at selected quantiles relative to the weight assigned to the median income, both when $k = 3, 4, 5, 6$ and at the limit when $k \rightarrow \infty$. The highest degree of upside inequality aversion occurs as $k \rightarrow \infty$, which corresponds to the mean income criterion.

4 | EMPIRICAL APPLICATIONS

4.1 | Distribution of household income in booms and busts

A large body of evidence suggests that inequality growth in the United Kingdom over the past few decades has been episodic and strongly related to the business cycle (see, e.g., Blundell & Etheridge, 2010). From 1993 onwards, the economy moved out of a recession and into a period of stable and moderate income growth across most of the income distribution. Then, from the late 1990s, a further rise in income occurred, largely concentrated at the upper part of the income distribution. The recession that followed the financial crisis in 2007/2008 led to sharp falls in incomes, especially at the upper part of the income distribution.

Our approach can be used to make unambiguous statements about the social welfare implications of these changes in the household income distribution. Our data come from the European Community Household Panel (ECHP) for 1995–2001, and from the European Union Statistics on Income and Living Conditions (EU-SILC) for 2005–2010.¹⁶ In each year, we restrict the sample to households with a male aged 25–64. We focus on the distribution of individual equivalent income, after adjusting for inflation and differences in household size and composition.¹⁷ In order to perform statistical inference, Appendix S2 develops distribution theory to test for upward and downward inverse stochastic dominance of any degree.¹⁸

Using our data, panel (a) of Figure 4 displays the evolution at different parts of the equivalent income distribution. To assess the changes in the distribution of individual equivalent income, we make pairwise yearly comparisons of all the distributions. Table 2 shows the ranking on the basis of second-degree dominance, denoting by “>” if the earlier year dominates, and by “<” if the later year dominates. We can see that 45 of a possible 91 pairwise yearly comparisons can be ranked on the basis of second-degree dominance. Furthermore, all but eight of these rankings are statistically significant at conventional levels. This still leaves us a long way short of a complete ranking but is nonetheless a useful first step.¹⁹ One insight from Table 2 is that any inequality averse social planner would conclude that social welfare is higher in 2007 than in the previous years. Another finding is that social welfare remains higher after the crisis as compared with 1994, our first year of observation.

¹⁶Unfortunately, these datasets do not provide information on income for the years 2002–2004.

¹⁷To adjust for differences in household size and composition, we use the OECD equivalence scale.

¹⁸We are not aware of asymptotic distribution theory for inverse stochastic dominance tests. Andreoli (2018) develops a test with pointwise confidence intervals, which essentially implies multiple testing across quantiles. In contrast, we represent the quantile function as a Gaussian continuous process that allows us to develop confidence bands. For alternative approaches to testing for standard stochastic dominance, see Abadie (2002), Anderson (1996), Barrett and Donald (2003), Linton et al. (2005), and Davidson and Duclos (2000), among others.

¹⁹The high rate of success in ranking income distributions by second-degree dominance contrasts with the findings in some other datasets (see Atkinson, 2008)

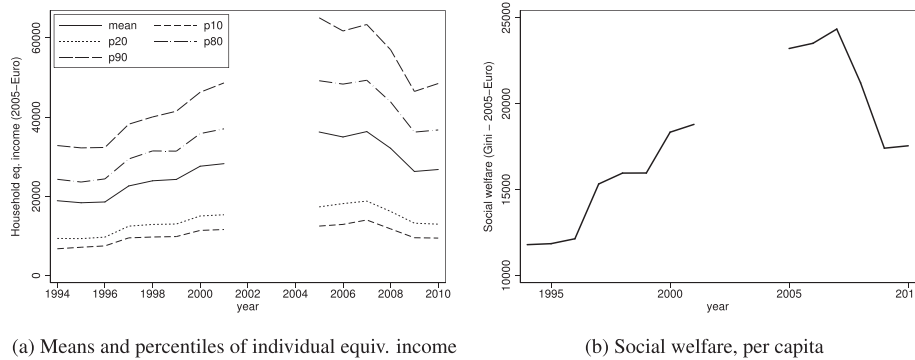


FIGURE 4 Time trends in the U.K. distribution of income and social welfare. *Note:* the data come from ECHP (1994–2001) and EU-SILC (2005–2010). We use the population weights supplied by EU-Stat. In panel (a), we show the evolution of the means and selected quantiles of the distributions of individual equivalent income, after adjusting for inflation and differences in household size and composition. In panel (b), we display the evolution of social welfare per capita based on the Gini social welfare function

Year	1995	1996	1997	1998	1999	2000	2001	2005	2006	2007	2008	2009	2010
1994			<*					<	<	<	<	<*	<
1995	<*	<						<	<	<	<	<*	
1996										<			
1997										<			
1998						<		<	<	<	<		
1999						<		<	<	<	<		
2000								<	<	<	<*		
2001								<	<*	<	<*		
2005										<*			>
2006										<			>
2007											>	>	>
2008													>
2009													

TABLE 2 Ranking of income distributions by second-degree dominance

Note: The table makes pairwise yearly comparisons of the income distributions over the period 1994–2010. We report the successful rankings based on second-degree dominance. We denote by “<” when the later year dominates the earlier year, and with “>” when the earlier year dominates the later year. We highlight the cases in which the level of statistical significance is below 0.05. * $p > 0.10$. ** $p > 0.05$.

In Table 3, we examine whether third-degree upward dominance raises the ranking success rate. We find that the use of this refinement matters little, if anything, for the ability to rank income distributions. By contrast, third-degree downward dominance provides an almost complete ranking of the income distributions. As shown in Table 4, this ranking criterion resolves all except one of the comparisons that were ambiguous under second-degree dominance. We can also see that these rankings are statistically significant at conventional levels.

Taken together, the findings in Tables 3 and 4 point to the importance of whether income differences between poorer individuals are viewed as more or less important for social welfare as compared with income differences between richer individuals. If the social planner is more concerned with income differences in the lower part of distributions, weaker criteria than third-degree upward dominance are required to make unambiguous conclusions about the changes in the distribution of income over the business cycle.²⁰ However, if the social planner focuses attention on income differences in the upper part of the distribution, as in the recent studies of the evolution of top incomes, a nearly complete ranking of income distributions can be achieved. In particular, it is then clear that social welfare steadily increased until 2007 and that the recession caused a reversion in social welfare to the level of year 2000. This can be seen clearly in panel (b) of Figure 4, which shows the estimated social welfare in each year as evaluated by the least restrictive social welfare function that ranks consistently with third-degree downward dominance (i.e., the Gini social welfare function). At its peak in 2007, the equally distributed equivalent income is above €24,000 per capita and the welfare loss due to inequality is about €11,400 per capita.

²⁰Table S1 shows the necessary degree of upward and downward dominance to achieve a complete ranking. The results show that the degree of upward dominance has to be quite high to raise the ranking success rate substantially.

TABLE 3 Ranking of income distributions by third-degree upward dominance

Year	1995	1996	1997	1998	1999	2000	2001	2005	2006	2007	2008	2009	2010
1994	<*	<*	<					<	<	<	<	<	<
1995		<	<					<	<	<	<	<	
1996										<			
1997											<		
1998					>*	<		<	<	<	<		
1999						<		<	<	<	<		
2000								<	<	<	<		
2001								<	<	<	<		
2005										<			>
2006										<			>
2007											>	>	>
2008													>
2009													

Note: The table makes pairwise yearly comparisons of the income distributions over the period 1994–2010. We report the successful rankings based on 3rd-degree upward inverse stochastic dominance. We denote by “<” when the later year dominates the earlier year, and with “>” when the earlier year dominates the later year. We highlight the cases in which the level of statistical significance is below 0.05. * $p > 0.10$. ** $p > 0.05$.

TABLE 4 Ranking of income distributions by third-degree downward dominance

Year	1995	1996	1997	1998	1999	2000	2001	2005	2006	2007	2008	2009	2010
1994	>		<	<	<	<	<	<	<	<	<	<	<
1995		<	<	<	<	<	<	<	<	<	<	<	<
1996			<	<	<	<	<	<	<	<	<	<	<
1997				<	<	<	<	<	<	<	<	<	<
1998					<**	<	<	<	<	<	<	<	<
1999						<	<	<	<	<	<	<	<
2000							<	<	<	<	<	>	>
2001								<	<	<	<	>	>
2005									>	<	>	>	>
2006										<	>	>	>
2007											>	>	>
2008												>	>
2009													<

Note: The table makes pairwise yearly comparisons of the income distributions over the period 1994–2010. We report the successful rankings based on 3rd-degree downward inverse stochastic dominance. We denote by “<” when the later year dominates the earlier year, and with “>” when the earlier year dominates the later year. We highlight the cases in which the level of statistical significance is below 0.05. * $p > 0.10$. ** $p > 0.05$.

4.2 | Evaluating the distributional effects of policy

To illustrate the usefulness of our approach for policy evaluations, we now apply it to Connecticut's Jobs First experiment.²¹ This randomized controlled trial assigned 2396 welfare recipients to Jobs First, whereas 4803 recipients were assigned to Aid for Dependent Children (AFDC). Compared with the high implicit tax rates and no time limit of the AFDC program, Jobs First expanded the earnings disregard and imposed a strict 21-month time limit on welfare participation. Under AFDC, the monthly earnings disregard was \$120 in the first year and \$90 thereafter, while statutory benefit reduction was 66% in the first 4 months, and 100% thereafter.²² In contrast, Jobs First entailed no benefit reduction below the federal poverty line and a 100% reduction beyond this.²³

Bitler et al. (2006) evaluated how the Jobs First program affected the distribution of earnings, transfers, and total income among participants. In line with the predictions from economic theory, the estimated quantile treatment effects reveal considerable heterogeneity in the impact of the program. To evaluate whether this program was an overall success, we

²¹For detailed information about the program and for descriptive statistics, we refer to Bloom et al. (2002) or Bitler et al. (2006).

²²Due to several expense disregards, lags in enforcement, and the implicit wage subsidy from the Earned Income Tax Credit, Bitler et al. (2006) estimate the effective benefit reduction at about 33%.

²³Compared with AFDC, Jobs First also expanded the work requirement, the asset limit, and transitional Medicaid, while enforcing stricter sanctions for violations (see Bloom et al., 2002).

	Upward dominance	Downward dominance
A. Ranking		
Degree of dominance	9	3
Dominating distribution	AFDC	Jobs First
B. Social welfare gain of dominating distribution		
ΔW_p	14.41%	10.94%
C. Weights at quantiles		
$p(0.05)$	89.39	1.90
$p(0.30)$	10.54	1.40
$p(0.70)$	0.03	0.60
$p(0.95)$	0.00	0.10

TABLE 5 Comparing income distributions under Jobs First and AFDC

Note: The results are based on the estimated quantile treatment effects reported in Bitler et al. (2006). Income is defined as the sum of earnings and transfers, averaged over quarters 1–16. Panel A shows the degree of upward and downward dominance necessary to rank the income distributions under Jobs First and AFDC. Panel B shows the gains in social welfare (in percent) in the dominating distribution. We report the change in social welfare for the least restrictive member of the parametric social welfare functions that rank consistently with the estimated degree of dominance. Panel C displays the weights at selected quantiles relative to the weight at the median income for the least restrictive member of the parametric social welfare functions.

extend on their analysis by using our framework to rank the actual and counterfactual income distributions and to quantify the difference in social welfare between the two distributions. We focus on total income including transfers and earnings averaged over 4 years following treatment assignment.

Table 5 displays the results. In panel A, we report the degree of upward and downward dominance necessary to rank the distributions of total income under Jobs First and AFDC. By identifying the least restrictive member of the parametric social welfare functions that rank consistently with the estimated degree of dominance, we can also compute the social welfare level of the dominating distribution as compared with the dominated distribution. Panel B reports the percentage increase in social welfare in the dominating distribution. To ease the interpretation of the social preferences underlying the dominance results, panel C illustrates the weight functions of the least restrictive members of the parametric social welfare functions. For brevity, we report the ratios of the weights of the median individual compared with the 5% poorest, the 30% poorest, the 30% richest, and the 5% richest.

We can see that a refinement of second-degree dominance is necessary to rank the income distributions under Jobs First and AFDC. The first column shows that we need a high degree of upward dominance to reach an unambiguous ranking. If the social planner is sufficiently averse to income differences in the lower part of the distributions, AFDC unambiguously provides higher social welfare than Jobs First. For instance, the least restrictive member of the parametric welfare functions that rank consistently with ninth-degree upward dominance assigns about 10 times as much weight to the 30th percentile compared with the median income. With such social preferences, Jobs First is estimated to reduce social welfare by 14.4%.

The second column confirms the ability of third-degree downward dominance to resolve comparisons that were ambiguous under second-degree dominance. If the social planner supports the principle of first-degree UPTS, the Jobs First distribution dominates the AFDC distribution. This implies that an unambiguous conclusion can be drawn with quite unrestrictive social preferences; for example, it is sufficient to assign 1.4 times as much weight to the 30 percentile compared with the median income. By applying the least restrictive member of the parametric welfare functions that ranks consistently with third-degree downward dominance, that is, the member with the most progressive weight structure, we estimate that Jobs First increases social welfare by almost 11%.

5 | CONCLUSION

Since the seminal contributions of Kolm (1969) and Atkinson (1970), second-degree dominance has become a widely accepted criterion for ranking distribution functions. But in many applications where the distribution functions intersect, a reasonable refinement of this criterion is necessary to attain an unambiguous ranking. Our paper contributes by providing a general rank-dependent theory to unambiguously order a given set of distribution functions and quantify the social welfare level of a dominating distribution as compared with a dominated distribution. The goal of our approach is

to avoid making arbitrary restrictions or parametric assumptions about social welfare functions and allow researchers to state the weakest set of assumptions needed to rank distributions according to social welfare.

Our approach is based on two complementary sequences of nested inverse stochastic dominance criteria. The first sequence includes the traditional inverse dominance criteria of third and higher degrees; it is called *upward dominance* because it aggregates the integrated inverse distribution function from below and therefore places more emphasis on differences that occur in the lower part of the distributions. The second sequence is novel and complements the traditional criteria by placing more emphasis on differences that occur in the upper part of the distribution; we call it *downward dominance* because it aggregates the integrated inverse distribution function from above. Because the sequences are hierarchical, the sensitivity to differences in the lower (upper) part of the distribution increases with the degree of upward (downward) dominance. The two sequences coincide at second-degree dominance, and thus both satisfy the Pigou–Dalton transfer principle.

For each sequence, we show equivalence in the ranking of distributions according to the dominance criteria and a general family of rank-dependent social welfare functions. Because the sequences of dominance criteria are nested, our equivalence results allow us to uniquely identify the largest subfamily of rank-dependent welfare functions—and thus the least restrictive rank-dependent social preferences—that give an unambiguous ordering of a given set of distribution functions. We also provide a characterization of the largest subfamily of rank-dependent social welfare functions that order consistently with dominance of any given degree. Because of the equivalence result, this characterization provides interpretation and justification not only for the social welfare functions but also for the use of higher degree dominance criteria in comparison of distribution functions. We further show that the members of two alternative parametric families of social welfare functions can be divided into subfamilies according to their relationship with the nested inverse stochastic dominance criteria. The parametric families are well known, easily implementable, and the estimated social welfare can be given an interpretation as equally distributed equivalent incomes.

We show the usefulness of our approach with two empirical applications. The first uses data from the United Kingdom to study how the distribution of household income evolved over a boom and a bust era in the British economy. We show how our framework can be used to make unambiguous statements about the social welfare implications of the changes in the household income distribution over the business cycle. The second uses random-assignment data to evaluate the distributional effects of Connecticut's Jobs First program, which involved generous earnings disregard and strict time limits. We use our framework to infer the least restrictive social preferences that allow an unambiguous conclusion of whether this program was an overall success. In both applications, we find that third-degree downward dominance is a particularly powerful refinement of second-degree dominance, providing an almost complete ranking of the distribution functions. By comparison, the traditional criterion of third-degree upward dominance resolves few of the comparisons that were ambiguous under second-degree dominance.

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OPEN RESEARCH BADGES



This article has been awarded Open Data Badge for making publicly available the digitally-shareable data necessary to reproduce the reported results. Data is available at <http://qed.econ.queensu.ca/jae/datasets/mogstad001/>.

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SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of the article.

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