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Compensated discrete choice and the Slutsky equation

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Abstract: Consumers often face choice settings in which alternatives are discrete. Examples include choices between variants of differentiated products, modes of urban transportation, residential locations, etc. In this paper compensated price elasticities and a corresponding (aggregate) Slutsky equation for discrete choice models are derived. A remarkable feature of compensated price elasticities in the discrete case is that they usually are not symmetric, as compensated elasticities with respect to a price increase versus a price decrease may be different. Finally, compensated marginal price effects and elasticities are derived for selected examples.

Keywords: Compensated choice, Discrete/continuous choice, Slutsky equation, Marginal compensated effects

JEL classification: C25, C43, D11

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Sammendrag

Tradisjonell mikroøkonomisk teori for konsumenters adferd forutsetter at varene som etterspørres er uendelig delbare. Med dette menes at konsumentene kan etterspørre et hvilket som helst kvantum av en vare. Det er imidlertid åpenbart at de fleste varer som etterspørres slett ikke er uendelig delbare. Eksempler er valg mellom varianter av differensierte produkter (slik som biler, kjøleskap, støvsuger, mobiltelefon), alternative transportalternativer, utdanning (retning og nivå), type jobb, yrkesdeltaking, bolig, osv. I slike tilfeller kan teorien for såkalte diskrete valg benyttes.

Mens tradisjonell mikroøkonomisk teori omfatter analyse av kompenserte priselastisiteter har det hittil ikke vært utviklet tilsvarende resultater innen diskret valghandlingsteori. Et sentralt resultat i tradisjonell teori er den såkalte Slutsky likningen, som spesifiserer sammenhengen mellom de kompenserte og ukompenserte priselastisitetene.

I denne artikkelen etableres kompenserte priselastisiteter for situasjoner med diskrete valg. Videre utledes en aggregert Slutsky-likning for diskrete valgmodeller (diskret Slutsky-likning). Denne har noen fellestrekk med Slutsky-likningen i tradisjonell teori, men har også noen egenskaper som avviker fra denne. Et bemerkelsesverdig trekk ved de kompenserte priselastisitetene i det diskrete tilfellet er at de kan være asymmetriske, da de kompenserte elastisitetene med hensyn til en prisøkning kontra en prisnedgang kan være ulike.

Endelig utledes diskrete Slutsky-likninger for utvalgte eksempler.

1 Introduction

The theory of compensated demand is well developed for the standard textbook case with infinitely divisible quantities of commodities: see, for example, Hausman (1981). However, in many instances consumers face choice settings where alternatives are discrete. These include choice between variants of a differentiated product, modes of urban transportation, residential location, types of education, types of child-care, etc. In the context of discrete choice settings, one cannot apply the standard microeconomic textbook approach to obtain individual demand functions. The reason is that the set of feasible consumption alternatives is not a continuum and the corresponding utility function is not differentiable. Therefore, the standard approach based on the usual marginal calculus technique does not apply. Instead, the operational concept that corresponds to individual demand in the infinitely divisible case is aggregate demand: that is, the compensated and uncompensated choice probabilities.

In the context of welfare analysis in discrete choice settings, the standard tools of applied welfare economics are not directly applicable either. Consequently, it is important to develop practical welfare measures in these settings, because welfare judgments are of major interest in applications with finite choice sets, as in the examples mentioned above. As in the textbook case, typical evaluations of interest are the welfare effect of changes in prices, taxes, quality attributes of alternatives and choice sets. In addition to welfare analysis, marginal compensated measures also serve to justify key price indexes.

A central aspect of welfare assessments is the calculation of marginal compensated price effects. In cases with infinitely divisible quantities of commodities, the Slutsky equation plays a key role. The Slutsky equation, referred to as the "fundamental equation in value theory" by Hicks (1936), allows one to compute the compensated price elasticities from the corresponding demands, uncompensated price and income elasticities.

The main purpose of this paper is to derive a Slutsky equation for random utility models of discrete choice behavior (discrete Slutsky equation). The Slutsky equation in this case is an aggregate one as it represents the relationship between the compensated and uncompensated price and income elasticities of the corresponding choice probabilities. The Slutsky equation we obtain also covers a specific case of discrete/continuous choice. In a separate paper (Dagsvik et al., 2021), marginal compensated wage elasticities for discrete labor supply models are obtained.

In the special case where the utility function is linear in income and additive separable in income and price, there are no income effects and the marginal compensated effects then equal the corresponding uncompensated effects. However, when utilities are non-linear in income one can no longer express marginal compensated effects by simple formulas. In such cases stochastic Monte Carlo (MC) simulation methods have previously been used to calculate welfare measures. Discrete choice models where utilities are non-linear in income are quite common in urban travel demand studies, see, for example, the empirical applications discussed by Ben-Akiva and Lerman (1985), and McFadden (1981), where alternative-specific costs are divided by monthly income.

Dagsvik and Karlström (2005) obtained analytic formulas for compensated choice probabilities in discrete choice models when utility is non-linear in income.[1] The results in Dagsvik and Karlström (2005) serve as a point of departure for deriving the Slutsky equation in the discrete case. It turns out that the discrete Slutsky equation is, in part, analogous to the Slutsky equation in the standard indivisible case, but it also differs in important ways. Specifically, a remarkable feature of the compensated marginal price effects in the discrete case is that they are usually not symmetric, as the marginal compensated price effects with respect to a price increase versus a price decrease in general may be different.

Instead of deriving analytic results for the Slutsky equation, one could alternatively conduct (MC) simulations of marginal compensated effects, as in McFadden (1999) and Niemeier (1997). However, it is important to derive analytic results for several reasons, provided that they are simple to use in practical computations, as is the case with the analytic results obtained in this paper. First, analytic expressions reveal key qualitative features of the marginal compensated effects, such as the property that the marginal effects with respect to a price increase versus a price decrease in general may be different, as mentioned above. Second, analytic results are more precise than results based on MC simulations, because they are not plagued with simulation errors.

An early welfare analysis for discrete/continuous choice was undertaken by Small and Rosen (1981). They seem to be the only ones who have previously discussed marginal compensated effects in this regard. Their analysis is, however, incomplete and, in part, misleading, as will be discussed further later.

The paper is organized as follows. In Section 2 we discuss the notion of compensated choice, expenditure function and corresponding choice probabilities in the discrete choice setting based on random utility representations. In Section 3 the analysis of Small and Rosen

[1] Further results regarding welfare analysis in the context of discrete choice have been obtained by Bhattacharya (2015, 2018).

(1981) is revisited. Section 4 deals with joint ex ante and ex post compensated choice and the corresponding choice probabilities. In Section 5 we discuss marginal compensated effects with special reference to the discrete Slutsky equation and in Section 6 we explain why the marginal compensated effects in the discrete case are asymmetric. Section 7 considers some selected special cases.

2. Compensated discrete choice

In discrete choice theory based on random utility representations, the notion of uncompensated and compensated demand is more complicated than in the conventional textbook case. In addition, separate analyses are necessary for the one-period setting versus the two-period setting: that is, before (ex ante) and after (ex post) a reform is introduced. As will be explained shortly, the reason for this is that in random utility models there is no unique deterministic correspondence between prices and expenditure, since utilities are random functions of prices and income.

Remember that the random utility representation is motivated by the fact that not all variables that influence preferences are observable to the researcher. Some of the variables affecting preferences of a consumer may even be random to the consumer himself. The reason is that tastes may vary in an unpredictable way from one moment to the next across identical choice settings due to psychological factors and the difficulties decision-makers have establishing a definitive ranking of the alternatives. Consequently, the utilities, the demand and the expenditure functions all become interdependent random functions. This feature necessitates a careful probabilistic analysis based on the corresponding joint distribution function of these random variables.

Consider a general setting in which the consumer faces a choice of a composite continuous good and a set of discrete (and possibly discrete/continuous) alternatives where the discrete alternatives are mutually exclusive. Let $U_j^*(x_0, x_j)$ be the utility of quantity x_0 of the composite good and let quantity x_j be associated with the discrete alternative *j*, $j = 1, 2,\dots, m$. Note that U_j^* depends on j because utility may depend on non-pecuniary attributes of alternative *j*. In the pure discrete case x_j is equal to 1 (when alternative *j* is chosen) or zero, but for the sake of comparison with Small and Rosen (1981) we shall also consider briefly the discrete/continuous case in which x_j takes values in $[0,\infty)$.

The consumer is assumed to maximize $U_j^*(x_0, x_j)$ subject to the budget constraint

$$x_0 + \sum_{j=1}^{m} p_j x_j = y, \;\; x_j \geq 0, \;\; x_j x_k = 0, \;\; k \neq j, \;\; \forall j,k,$$

where y denotes income and p_j the price of the discrete alternative j. The price of the composite infinitely divisible good with quantity x_0 is normalized to 1. Let $U_j(p_j, y)$ be the conditional indirect utility given the discrete alternative j: that is, $U_j(p_j, y)$ is the maximum of $U_j^*(x_0, x_j)$ subject to the budget constraint $x_0 + p_j x_j = y$. In the pure discrete case, where $x_j = 1$ the indirect utility conditional on alternative j admits the form $U_j(p_j, y) = U_j^*(y - p_j, 1)$. The general formulation above covers several (but not all) cases of interest. Other formulations appear in Dubin and McFadden (1984), Hanemann (1984) and Dagsvik (1994). Consider, for example, the choice of working in different labor market sectors, where it is understood that hours of work are fixed and possibly sector-specific. In this case of sectoral labor supply without taxes and with fixed hours of work, the conditional indirect utility can be expressed as $U_j(p_j, y) = U_j^*(y + w_j, 1)$ where now $p_j = w_j$ denotes the wage rate. Thus, the function $U_j(p_j, y)$ can be both increasing (occupational mobility and labor supply) and decreasing in p_j.

Assumption 1

The (conditional indirect) utility function has the structure $U_j(p_j, y) = v_j(p_j, y) + \varepsilon_j$, *where* $v_j(p_j, y)$ *is a deterministic function that is continuously differentiable, strictly increasing in y and strictly monotone in* p_j *and* $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ *is a stochastic vector with joint c .d .f. F that possesses a continuous probability density.*

Under Assumption 1 and with F being a multivariate extreme value distribution with Gumbel marginals, then the implied choice model becomes the Generalized Extreme Value (GEV) model (McFadden, 1978). Recall that the additive random utility structure assumed above, which is the same set-up as in Dagsvik and Karlström (2005), represents no loss of generality. Dagsvik (1994, 1995) and Joe (2001) have demonstrated that any random utility model can be approximated arbitrarily closely by a GEV model. Since the GEV family is a subclass of the random utility models generated by Assumption 1, it follows that Assumption 1 represents no essential loss of generality.

The agent's choice set of available alternatives may be a subset of this universal set, $\{1, 2, ..., m\}$. However, for simplicity we assume formally that all alternatives are available but with price $p_j = \infty$, $v_j(\infty, y) = v_j(p_j, 0) = -\infty$ and $\partial v_j(p_j, y) / \partial p_j = 0$ if alternative *j* is not available to the consumer. Evidently, this represents no loss of generality. Also, in the following (under Assumption 1), we shall sometimes write v_j or $v_j(y)$ instead of $v_j(p_j, y)$. Let $J(p, y)$ be the (Marshallian) choice function where $p = (p_1, p_2, ..., p_m)$ is the vector of prices: that is, $J(p, y) = j$ if the discrete alternative *j* is chosen, given prices and income (*p*, *y*).

Let $V(p, y)$ be the indirect utility defined by $V(p, y) = \max_k U_k(p_k, y)$ and define the conditional expenditure function $e_j(p_j, u)$, given alternative *j*, by the relation

$$u = U_j(p_j, e_j(p_j, u)) = v_j(p_j, e_j(p_j, u)) + \varepsilon_j$$

where *u* is a given utility level. When $v_j(p_j, y)$ is strictly increasing in *y* it follows that $e_j(p_j, u)$ is uniquely determined. The expenditure function (unconditional) $e(p, u)$ is therefore given by

(2.1) $$e(p, u) = \min_{j \le m} e_j(p_j, u)$$

which is equivalent to

$$V(p, e(p, u)) = u.$$

Since the utility function depends on random taste variables the expenditure function becomes stochastic. Let $J^c(p, u)$ denote the compensated discrete choice function given prices and utility level (*p*, *u*) and let $x_j^c(p, u)$ be the corresponding conditional compensated demand given alternative *j*. We then have that

(2.2) $$J^c(p, u) = J(p, e(p, u)) \quad \text{and} \quad x_j^c = x_j^c(p_j, u) = x_j(p_j, e_j(p_j, u)).$$

Similarly, we have that

(2.3) $$J(p, y) = J^c(p, V(p, y)) \quad \text{and} \quad x_j(p, y) = x_j^c(p, V(p, y)).$$

Next, we consider the corresponding aggregate demand functions which in this context are the choice probabilities. Let

$$\varphi_j(p, y) = P(J(p, y) = j) = P(v_j(p_j, y) + \varepsilon_j = \max_r (v_r(p_r, y) + \varepsilon_r))$$

and define the social surplus function

$$H(v_1, v_2, ..., v_m) = E \max_k (v_k + \varepsilon_k)$$

$$= \int_0^{\infty} (1 - F(u - v_1, u - v_2, ..., u - v_m)) du - \int_{-\infty}^{0} F(u - v_1, u - v_2, ..., u - v_m) du$$

which exists provided that the expectations of the random taste shifters exist. The definition of the social surplus function above can be extended to the case where the expectations of the random taste shifters do not exist, see McFadden (1981). It follows from Assumption 1 and McFadden (1981, pp. 212–214) that

(2.4) $$\varphi_j(p, y) = H_j'(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y))$$

$$= \int_{-\infty}^{\infty} F_j'(u - v_1(p_1, y), u - v_2(p_2, y), ..., u - v_m(p_m, y)) du$$

where F_j' denotes the partial derivative of F with respect to component j. Note that the relation in (2.4) differs from Roy's identity when utility is non-linear in income. We call $\varphi_j(p, y)$ the Marshallian (or uncompensated) probability of choosing alternative j. The corresponding uncompensated conditional demand function given the discrete alternative j follows from Roy's identity, that is,

(2.5) $$x_j = x_j(p_j, y) = -\frac{v_{j1}'(p_j, y)}{v_{j2}'(p_j, y)}$$

where v_{jk}', $j = 1, 2, \ldots, m$, $k = 1, 2$, denotes the derivative with respect to component k. We note that the conditional demand functions in (2.5) are deterministic since they depend only on the deterministic terms of the utility function.

Similarly to the uncompensated demands the concept that corresponds to (aggregate) compensated demand for the discrete alternatives is the compensated (or Hicksian) choice probability. It is defined as

$$P_j^c(p, u) = P(J^c(p, u) = j) = P(e_j(p_j, u) = e(p, u)).$$

Dagsvik and Karlström (2005, Theorems 1 and 2) have derived the formula for $P_j^c(p, u)$ and the distribution of $e(p, u)$ under Assumption 1, which are given by

$$P_j^c(p, u) = \int_0^{\infty} F_j'(u - v_1(p_1, y), u - v_2(p_2, y), ..., u - v_m(p_m, y)) v_j(p, dy)$$

and

$$P(e(p, u) \le y) = 1 - F(u - v_1(p_1, y), u - v_2(p_2, y), ..., u - v_m(p_m, y)).$$

3. The analysis of Small and Rosen revisited

Small and Rosen were the first to carry out welfare analysis in settings with discrete/continuous choice in their seminal paper (Small and Rosen, 1981) (SM). On pp. 105-114 in SM the analysis parallels conventional deterministic microeconomic theory for infinitely divisible goods, albeit with important modifications. It is not until p. 115 that they introduce the notion of choice probability to deal with randomness in preferences. In SM (3.19), p. 116, they define uncompensated and compensated choice probabilities. Unfortunately, they do not discuss explicitly how these probabilities are to be interpreted in terms of standard random utility formulations. Accordingly, the relation given in SM (3.25) is not correct (or at best misleading). If we use our notation the analogue to SM (3.19) would be

$$P_j^c(p,u) = \varphi_j(p, e(p,u)).$$

However, the relation above is meaningless because the right side is random since $e(p,u)$ is a random function. Moreover, because the expenditure function is random it is only differentiable within intervals with stochastic boundaries. Consequently, SM (3.25), and the following equations in SM, p. 117, is not consistent with the usual random utility setting. In order to calculate changes in compensated demand under a price change from p to $\tilde{p}$ it would be necessary to start with the following aggregate compensated change given by

$$P(J^c(\tilde{p}, V(p,y)) = j) - P(J(p,y) = j),$$

which is far from straightforward to calculate. This is the task to be addressed in the next section.

4. Joint ex ante choice and ex post compensated choice

Although (2.4) and (2.5) may have some theoretical interest, they are not directly useful in contexts where one wishes to analyze compensated demand after a specific policy reform has been introduced. The reason is that one needs to account for the fact that utilities are stochastic and unobservable. Let $\tilde{p}$ denote the ex post price vector and p and y the ex ante price vector and income. It follows that the ex post compensated choice is $J^c(\tilde{p}, V(p,y))$. In the context of compensated demand it is assumed that the stochastic terms of the utility function remain invariant under the policy reforms. This assumption is common in welfare analysis based on random utility models. Unfortunately, it is in general the case that

$$P(J^c(\tilde{p},u) = j \mid V(p,y) = u) \neq P(J^c(\tilde{p},u) = j) = P_j^c(\tilde{p},u)$$

Because $J^c(\tilde{p},u)$ and $V(p,y)$ are interdependent random variables. Consequently, $P(J^c(\tilde{p},V(p,y))=j)$ may in general be different from $EP_j^c(\tilde{p},V(p,y))$. The latter formula would hold if $J^c(\tilde{p},u)$ and $V(p,y)$ were independent, which is not the case.

Before we turn to the challenge to derive the distribution of the ex post compensated choice $J^c(\tilde{p},V(p,y))$ we need some additional concepts. Define Y_j and Y by

$$Y_j = Y_j(\tilde{p}_j; p, y) = e_j(\tilde{p}_j, V(p,y))$$

and

$$Y = Y(\tilde{p}; p, y) = \min_{j\le m} Y_j(\tilde{p}_j; p, y).$$

Whereas Y_j is the conditional ex post expenditure function given alternative *j*, Y is the unconditional ex post expenditure function that yields the income required to maintain the ex post utility level equal to the ex ante utility level.

Next, we consider the joint probability of the ex ante and ex post choice under a price change. Let $Q^c(j,k)=Q^c(j,k;\tilde{p},p,y)$ be the joint compensated probability of choosing alternative *j* ex ante and alternative *k* ex post (which means that the respective utility levels of the chosen alternatives before and after the reform are the same). Thus,

$$Q^c(j,k;\tilde{p},p,y) = P(U_j(p_j,y)=\max_r U_r(p_r,y)=U_k(\tilde{p}_k,Y)=\max_r U_r(p_r,Y)).$$

For simplicity we shall, most of the time, write $Q^c(j,k)$ instead of $Q^c(j,k;\tilde{p},p,y)$. Let

$$Q_k^c = Q_k^c(\tilde{p},p,y) = \sum\nolimits_r Q^c(r,k) = P(Y_k(\tilde{p}_k;p,y)=Y(\tilde{p};p,y))$$

which is the probability of choosing alternative *j* ex post conditional on the ex post utility being equal to the ex ante utility. Let y_j be determined by $v_j(p_j,y)=v_j(\tilde{p}_j,y_j)$, that is, y_j is the ex post income that ensures that the deterministic parts of the ex ante utility and ex post utility of alternative *j* are equal. If alternative *j* belongs to the ex ante choice set but not the ex post choice set, $y_j=\infty$. If alternative *j* belongs to the ex post choice set but not the ex ante choice set, $y_j=0$.

Theorem 1

Under Assumption 1 the compensated choice probability of choosing alternative j ex ante and alternative k ex post is given by

(4.1) $$Q^c(j,k) = -1\{y_j>y_k\}\int_{y_k}^{y_j} H''_{jk}(\psi_1(z),\psi_2(z),...,\psi_m(z))v_k(\tilde{p}_k,dz)$$

$$= 1\{y_j > y_k\} \int_{y_k}^{y_j} \int_{-\infty}^{\infty} F''_{jk}(u - \psi_1(z), u - \psi_2(z), ..., u - \psi_m(z)) du v_k(\tilde{p}_k, dz)$$

when $k \neq j,\ j, k > 0$, *where* $\psi_r(z) = \max(v_r(p_r, y), v_r(\tilde{p}_r, z))$ *for all r*. *Furthermore, when j = k, then*

(4.2) $$Q^c(j, j) = H'_j(\psi_1(y_j), \psi_2(y_j), ..., \psi_m(y_j))$$

$$= \int_{-\infty}^{\infty} F'_j(u - \psi_1(y_j), u - \psi_2(y_j), ..., u - \psi_m(y_j)) du.$$

The formulas in Theorem 1 can be used to compute the ex post compensated joint choice probabilities $\{Q^c(j, k)\}$ and the ex post compensated probabilities $\{Q_k^c\}$ after a change of the price vector from p to $\tilde{p}$. Theorem 1 is equivalent to Theorem 4 in Dagsvik and Karlström (2005).[2] In this paper we provide a simplified proof of Theorem 1 in the appendix. Note that $Q^c(j, j)$ given in (4.2) has the same structure as a choice probability, that is, the probability of choosing alternative *j* when the utility of alternative *r* equals $\psi_r(y_j) + \varepsilon_r$, $r = 1, 2, \ldots, m$.

Consider the case when $v_r(p_r, y) = v_r^*(y - p_r)$ for all *r* where v_r^* is an increasing function. Then, under Assumption 1 it follows, for $j \neq k$, that

$$Q^c(j,k) = \{y_j > y_k\} \int_{y_k}^{y_j} \frac{\partial H'_j(\psi_1(z), \psi_2(z), ..., \psi_m(z)) dz}{\partial p_k}.$$

Thus, we realize that the integrand of the integral in the expression for $Q^c(j, k)$ in this case can be obtained by integrating the Marshallian choice probabilities as functions of $\{\psi_r(z)\}$ with respect to *z*.

The result of Theorem 1 is instrumental in deriving compensated marginal effects and the corresponding Slutsky equation, as addressed in the next section.

5. The discrete Slutsky equation

We start with a brief review of the Slutsky equation in standard consumer theory where the quantities of goods are assumed to be infinitely divisible. Let $d_k(p, y)$ denote the (Marshallian)

[2] There is an error in Theorem 4 in Dagsvik and Karlström (2005). Eq. (4.2) in Theorem 1 corrects the corresponding expression in Dagsvik and Karlström (2005).

demand of commodity *k* as a function of prices and income (*p*, *y*) and let $d_k^c(p,u)$ denote the corresponding compensated demand function. The compensated demand function is not directly observable because it depends on the unobservable utility level. However, Slutsky (1915) showed how the marginal compensated demands can be obtained from the corresponding marginal Marshallian demands through the so-called Slutsky equation, which is given by

(5.1) $$\partial d_j^c(p,u)/\partial p_k = \partial d_j(p,y)/\partial p_k\,|_{y=e(p,u)} + d_k(p,e(p,u))\partial d_j(p,y)/\partial y\,|_{y=e(p,u)}$$

where $e(p,u)$ is the expenditure function.[3] This equation allows one to compute the unobserved marginal compensated demands with respect to price changes by using the corresponding marginal Marshallian demands (Varian, 1992). The equation in (5.1) can also be expressed as

(5.2) $$\partial d_j^c(p,u)/\partial p_k\,|_{u=V(p,y)} = \partial d_j(p,y)/\partial p_k + d_k(p,y)\partial d_j(p,y)/\partial y$$

where $V(p,y)$ is the indirect utility function. The advantage of (5.2) compared to (5.1) is that the marginal compensated effect can be evaluated in observable terms, that is, prices and income.

Consider next the discrete case. Define

$$\frac{\partial^+\varphi_j^c}{\partial p_k} = \lim_{\tilde{p}_k \downarrow 0} \frac{Q_j^c(\tilde{p},p,y)-\varphi_j(p,y)}{\tilde{p}_k - p_k} \quad \text{and} \quad \frac{\partial^-\varphi_j^c}{\partial p_k} = \lim_{\tilde{p}_k \uparrow 0} \frac{Q_j^c(\tilde{p},p,y)-\varphi_j(p,y)}{\tilde{p}_k - p_k}.$$

The expressions above are the right and left derivatives of $Q_j^c(\tilde{p},p,y)$ with respect to the ex post price $\tilde{p}_k$, evaluated at p_k. They correspond to the right and left marginal compensated effects of the probability of choosing alternative *j* resulting, respectively, from an increase or decrease of the price of alternative *k*. As we shall see below, it turns out that in general one has $\partial^+\varphi_j^c/\partial p_k \neq \partial^-\varphi_j^c/\partial p_k$, which means that in general the derivative $\partial\varphi_j/\partial p_k$ does not exist.

Theorem 2 (discrete Slutsky equation)

Assume that Assumption 1 holds with $v_j(p_j,y)$ *strictly decreasing in* p_j *for all j. Then*

$$\frac{\partial^+\varphi_j^c}{\partial p_j} = \frac{\partial\varphi_j}{\partial p_j} + x_j \cdot \frac{\partial\varphi_j}{\partial y}, \qquad \frac{\partial^-\varphi_j^c}{\partial p_j} = \frac{\partial\varphi_j}{\partial p_j},$$

[3] Usually, the uncompensated marginal price effect appears on the left side in the Slutsky equation. We prefer to write the compensated marginal price effect on the left side, in order to be consistent with the discrete Slutsky equation.

$$\frac{\partial^+\varphi_j^c}{\partial p_k} = \frac{\partial \varphi_j}{\partial p_k} \cdot \frac{\partial v_j / \partial y}{\partial v_k / \partial y} \quad \textit{and} \quad \frac{\partial^-\varphi_j^c}{\partial p_k} = \frac{\partial \varphi_j}{\partial p_k}$$

for $k \neq j$, *where*

$$x_j = -\frac{\partial v_j / \partial p_j}{\partial v_j / \partial y}.$$

The proof of Theorem 2 is given in the appendix.

Note that x_j is given by Roy's identity applied to the systematic part $v_j(p_j, y)$ of the utility function. In the context of discrete/continuous choice settings x_j can therefore be interpreted as conditional demand, given the choice *j*. Furthermore that only the equation determining $\partial^+\varphi_j^c / \partial p_j$ is similar to the standard Slutsky equation in (5.1) with income effect given by $-x_j \partial\varphi_j / \partial y$. Note also that the difference between the direct left marginal compensated effect minus the direct right marginal compensated effect is equal to the income effect.

From Theorem 2 the next result is immediate.

Corollary 1

Under the assumptions of Theorem 2 there is no income effect associated with a price decrease.

In Section 6 we discuss why the asymmetry in the Slutsky equation occurs in discrete choice settings. In fact, it becomes clear from our analysis that Corollary 1 holds even under weaker assumptions than Assumption 1.

We noted above that in some cases, such as in models of labor supply and occupational mobility, the utility function is increasing in "prices" (wage rates). By symmetry the result in the next corollary follows readily from Theorem 2.

Corollary 2

Under Assumption 1 with $v_j(p_j, y)$ *strictly increasing in* p_j *for all j we have that*

$$\frac{\partial^-\varphi_j^c}{\partial p_j} = \frac{\partial \varphi_j}{\partial p_j} - x_j \cdot \frac{\partial \varphi_j}{\partial y}, \qquad \frac{\partial^+\varphi_j^c}{\partial p_j} = \frac{\partial \varphi_j}{\partial p_j},$$

$$\frac{\partial^- \varphi_j^c}{\partial p_k} = \frac{\partial \varphi_j}{\partial p_k} \cdot \frac{\partial v_j / \partial y}{\partial v_k / \partial y} \quad and \quad \frac{\partial^+ \varphi_j^c}{\partial p_k} = \frac{\partial \varphi_j}{\partial p_k}$$

for $k \neq j$, *where*

$$x_j = \frac{\partial v_j / \partial p_j}{\partial v_j / \partial y}.$$

From Theorem 2 the next corollary is also immediate.

Corollary 3

Under Assumption 1 with $v_j(p_j, y) = v^*(y - p_j)$ *for some strictly increasing function* $v^*(\cdot)$ *that is independent of j, we have that*

$$\frac{\partial^+ \varphi_j^c}{\partial p_j} = \frac{\partial \varphi_j}{\partial p_j} + \frac{\partial \varphi_j}{\partial y}, \quad \frac{\partial^- \varphi_j^c}{\partial p_j} = \frac{\partial \varphi_j}{\partial p_j},$$

$$\frac{\partial^+ \varphi_j^c}{\partial p_k} = \frac{\partial \varphi_j}{\partial p_k} \frac{\partial v^*(y - p_j) / \partial y}{\partial v^*(y - p_k) / \partial y} \quad and \quad \frac{\partial \varphi_j}{\partial p_k} = \frac{\partial^- \varphi_j^c}{\partial p_k}$$

for $k \neq j$.

As mentioned above, the case of discrete labor supply models (e.g. van Soest, 1995, and Dagsvik and Strøm, 2006) is analyzed in a separate paper (Dagsvik et al., 2021). The reason is that this case does not immediately fit into the framework considered here because the price (wage rate) is the same for all discrete alternatives (positive hours of work schedules).

Consider next the discrete/continuous case where the conditional demand functions are determined by Roy's identity, as explained above. Let $\bar{X}_j = x_j \varphi_j$, where x_j is given by (2.5). That is, $\bar{X}_j$ is the unconditional (aggregate) demand for alternative j in the case of discrete/continuous choice. For the conditional demands the direct marginal effect will obviously satisfy the conventional Slutsky equation: that is,

$$\frac{\partial x_j^c}{\partial p_j} = \frac{\partial x_j}{\partial p_j} + x_j \cdot \frac{\partial x_j}{\partial y} \quad \text{and} \quad \frac{\partial x_j^c}{\partial p_k} = 0$$

for $k \neq j$, since x_j does not depend on p_k. We therefore obtain the relation

$$\frac{\partial^{\pm} \bar{X}_j^c}{\partial p_k} = x_j \cdot \frac{\partial^{\pm} \varphi_j^c}{\partial p_k} + \varphi_j \cdot \frac{\partial x_j^c}{\partial p_k}.$$

Thus, the next corollary follows readily from Theorem 2.

Corollary 4

Under Assumption 1 with $v_j(p_j, y)$ *strictly decreasing in* p_j *it follows that*

$$\frac{\partial^+ \bar{X}_j^c}{\partial p_j} = \frac{\partial \bar{X}_j}{\partial p_j} + x_j \cdot \frac{\partial \bar{X}_j}{\partial y}, \quad \frac{\partial^- \bar{X}_j^c}{\partial p_j} = \frac{\partial \bar{X}_j}{\partial p_j} + \bar{X}_j \cdot \frac{\partial x_j}{\partial y},$$

$$\frac{\partial^+ \bar{X}_j^c}{\partial p_k} = \frac{\partial \bar{X}_j}{\partial p_k} \cdot \frac{\partial v_j / \partial y}{\partial v_k / \partial y} \quad \textit{and} \quad \frac{\partial^- \bar{X}_j^c}{\partial p_k} = \frac{\partial \bar{X}_j}{\partial p_k}$$

for $k \neq j$, *where*

$$x_j = -\frac{\partial v_j / \partial p_j}{\partial v_j / \partial y}.$$

We note that, as discussed in Section 3, the relations in Corollary 4 differ in important ways from the corresponding (misleading) relations given in Small and Rosen (1981, pp. 116–118).

Quality changes

In Theorems 1 and 2 only compensated effects triggered by price changes are accounted for. However, in many contexts it is of interest to calculate the compensated effects under quality changes. For example, we may wish to evaluate the effects of changes in travel times in urban transportation mode choice models. It is easy to extend the results of Theorem 1 to accommodate changes in non-pecuniary attributes of the discrete alternatives. To make the following discussion precise, let

$$U_j(p_j, t_j, y) = v_j(p_j, t_j, y) + \varepsilon_j$$

be the utility function in the extended setting where t_j is a vector of non-pecuniary attributes (such as travel times) specific to alternative *j*. In this extended choice setting define $\psi_j^*(z) = \max(v_j(p,t,y), v_j(\tilde{p}, \tilde{t}, z))$ where $\tilde{t}_j$ denotes the ex post vector of non-pecuniary attributes. The modified version of (4.1) is obtained by replacing $\{y_k\}$, $\{\psi_k(z)\}$ and $v_j(\tilde{p}_j, z)$ with $\{y_k^*\}$, $\{\psi_k^*(z)\}$ and $v_j(\tilde{p}_j, \tilde{t}_j, dz)$ respectively, where y_k^* is determined by $v_k(p,t,y) = v_k(\tilde{p}, \tilde{t}, y_k^*)$. Similarly, (4.2) is to be modified by replacing $\{\psi_k(y_j)\}$ with $\{\psi_k^*(y_j^*)\}$.

Consider next the marginal compensated effects with respect to the non-pecuniary attributes in the case where they are continuous variables.

Corollary 5

Assume that the utility function is given by $U_j(p_j,t_j,y)=v_j(p_j,t_j,y)+\varepsilon_j$ *where* $v_j(p_j,t_j,y)$ *is strictly increasing in y and strictly decreasing in* p_j *for all j and* $t_j=(t_{j1},t_{j2},...)$*. Then*

$$\frac{\partial^+\varphi_j^c}{\partial t_{js}}=\frac{\partial\varphi_j}{\partial t_{js}}-\frac{\partial v_j/\partial t_{js}}{\partial v_j/\partial y}\cdot\frac{\partial\varphi_j}{\partial y},\quad \frac{\partial^-\varphi_j^c}{\partial t_{js}}=\frac{\partial\varphi_j}{\partial t_{js}},$$

$$\frac{\partial^+\varphi_j^c}{\partial t_{ks}}=\frac{\partial\varphi_j}{\partial t_{ks}}\cdot\frac{\partial v_j/\partial y}{\partial v_k/\partial y}\quad \text{and}\quad \frac{\partial^-\varphi_j^c}{\partial t_{ks}}=\frac{\partial\varphi_j}{\partial t_{ks}}$$

for $k\neq j$.

6. Why are compensated effects asymmetric?

To get some intuition to why the left marginal compensated effect in general may be different from the right marginal effect in the discrete case we shall discuss the binary case with two alternatives, 1 and 2. The argument in the general case with many alternatives is similar. The result that the marginal effects are asymmetric does not depend on the utility structure being separable (additive or multiplicative) in a systematic (deterministic) term and random term. To realize this, we now relax Assumption 1 by assuming that the utility $U_j(p_j,y)$ is a random function that is strictly decreasing in price p_j, $j=1,2,$ and strictly increasing in income *y* with probability 1. Let y_j be determined by $U_j(p_j,y)=U_j(\tilde{p}_j,y_j)$ and let Y_{jk} be the ex post expenditure when the ex ante choice is *j* and the ex post choice is *k*, *j*, *k* = 1, 2. Clearly, $y_j=Y_{jj}$ is deterministic due to the fact that $U_2(p_2,y)$ is almost surely strictly increasing in *y*. However, Y_{jk} may be stochastic when $k\neq j$.

Consider first the case where the price of alternative 2 decreases to $\tilde{p}_2<p_2$, whereas the price of alternative 1 remains unchanged. When the ex ante choice is alternative 1 and the ex post choice is alternative 2 then there are individual income effects. However, it turns out that the compensated probability $Q^c(1,2)$ becomes equal to the corresponding uncompensated probability. To realize this, note that

$$(6.1)\qquad Q^c(1,2) = P(U_2(p_2,y) < U_1(p_1,y) = U_2(\tilde{p}_2,Y_{12}) > U_1(p_1,Y_{12}))$$

which implies that the income compensation $Y_{12} - y$ is stochastic and $Y_{12} < y$ almost surely because $U_1(p_1,y) > U_1(p_1,Y_{12})$. Therefore, $Q^c(1,2)$ reduces to

$$(6.2)\qquad \begin{aligned} Q^c(1,2) &= P(U_2(p_2,y) < U_1(p_1,y) = U_2(\tilde{p}_2,Y_{12})) \\ &= P(U_2(p_2,y) < U_1(p_1,y) < U_2(\tilde{p}_2,y)) \\ &= P(U_1(p_1,y) < U_2(\tilde{p}_2,y)) - P(U_1(p_1,y) < U_2(p_2,y)) = \varphi_2(p_1,\tilde{p}_2,y) - \varphi_2(p,y) \end{aligned}$$

which is the uncompensated probability of choosing alternative 1 ex-ante and alternative 2 ex-post, which is equal to the difference between the ex post and ex ante probability of choosing alternative 2. In other words, the income compensation does not influence the joint choice probability $Q^c(1,2)$. Furthermore

$$Q^c(2,2) = P(U_1(p_1,y) < U_2(p_2,y) = U_2(\tilde{p}_2,y_2) > U_1(p_1,y_2))$$

which implies that the income compensation is deterministic and $y_2 < y$. Thus, $Q^c(2,2)$ reduces to

$$(6.3)\qquad Q^c(2,2) = P(U_1(p_1,y) < U_2(p_2,y)) = \varphi_2(p,y)$$

which is the ex ante probability of choosing alternative 2. Hence it follows from (6.2) and (6.3) that

$$(6.4)\qquad Q_2^c = Q^c(1,2) + Q^c(2,2) = P(U_2(\tilde{p}_2,y) > U_1(p_1,y)) = \varphi_2(p_1,\tilde{p}_2,y)$$

which is the ex post uncompensated probability of choosing alternative 2. Thus, there are no income effects in this case at the aggregate level.

Consider next the case where $\tilde{p}_2 > p_2$ whereas the price of alternative 1 is kept fixed. In this case

$$(6.5)\qquad Q^c(1,2) = P(U_2(p_2,y) < U_1(p_1,y) = U_2(\tilde{p}_2,Y_{12}) > U_1(p_1,Y_{12})) = 0$$

because it is not possible that both $U_2(p_2,y) < U_2(\tilde{p}_2,Y_{12})$ and $U_1(p_1,y) > U_1(p_1,Y_{12})$.

Moreover,

$$Q^c(2,2) = P(U_1(p_1,y) < U_2(p_2,y) = U_2(\tilde{p}_2,y_2) > U_1(p_1,y_2))$$

which implies that the income compensation is deterministic and $y_2 > y$. Consequently,

$$(6.6)\qquad Q_2^c = Q^c(1,2) + Q^c(2,2) = Q^c(2,2) = P(U_2(p_2,y) > U_1(p_1,y_2)).$$

We note that there is an essential difference between the probabilities given in (6.4) and (6.6) because when the price of alternative 2 decreases the ex post compensated probability of choosing alternative 2 becomes equal to the ex post uncompensated probability of choosing alternative 2. In contrast, when the price increases there must be income compensation $y_2 - y$ in order to maintain the utility level.

To gain further intuition, assume next that Assumption 1 holds and let *G* be the c. d. f. of $\varepsilon_1 - \varepsilon_2$. Then y_2 is determined by $v_2(p_2, y) = \tilde{v}_2(\tilde{p}_2, y_2)$. It follows from (6.4) that

(6.7) $$\frac{\partial^- \varphi_2^c(p,y)}{\partial p_2} = \frac{\partial \varphi_2(p,y)}{\partial p_2} = G'(v_2(p_2,y) - v_1(p_1,y))\frac{\partial v_2(p_2,y)}{\partial p_2}$$

and from (6.6) that

(6.8) $$\frac{\partial^+ \varphi_2^c(p,y)}{\partial p_2} = -G'(v_2(p_2,y) - v_1(p_1,y)) \cdot \frac{\partial v_1(p_1,y)}{\partial y} \cdot \frac{\partial y_2}{\partial p_2}$$

$$= G'(v_2(p_2,y) - v_1(p_1,y)) \cdot \frac{\partial v_1(p_1,y)}{\partial y} \cdot \frac{\partial v_2(p_2,y)/\partial p_2}{\partial v_2(p_2,y)/\partial y}.$$

From (6.7) and (6.8) it follows that

$$\frac{\partial^+ \varphi_2^c(p,y)/\partial p_2}{\partial^- \varphi_2^c(p,y)/\partial p_2} = \frac{\partial v_1(p_1,y)/\partial y}{\partial v_2(p_2,y)/\partial y}$$

which is different from 1 provided that $v_j(p_j, y)$ is not additively separable in price and income.

7. Special cases

In this section we illustrate the theoretical results obtained above by calculating the marginal compensated effects in selected examples.

Example 1: Urban travel demand

This example is typical for applications in urban travel demand analysis, see McFadden (1981), and Ben-Akiva and Lerman (1985). The utility function has the form

$$U_j = v_j(p_j, t_j, y) + \varepsilon_j = \alpha_j + \beta \frac{p_j}{y} + \gamma_1 t_{j1} + \gamma_2 t_{j2} + \varepsilon_j, \;\; \alpha_1 = 0,$$

where the income variable *y* is the monthly wage, p_j is the cost of alternative *j*, t_{j1} is “on-vehicle time” of alternative *j*, t_{j2} is “out-of-vehicle time” of alternative *j*, and $\{\alpha_j\}$, β, γ_1, and γ_2 are unknown parameters. When the random error terms are independent with Gumbel c. d. f. $\exp(-\exp(-x))$ for real *x* it follows that the choice model becomes the familiar multinomial logit model given by

(7.1) $$\varphi_j = \varphi_j(p,t,y) = \frac{\exp(\alpha_j + \beta p_j / y + \gamma_1 t_{j1} + \gamma_2 t_{j2})}{\sum_k \exp(\alpha_k + \beta p_k / y + \gamma_1 t_{k1} + \gamma_2 t_{k2})}.$$

Below we give the compensated elasticities with respect to travel costs and travel times based on (7.1). Specifically, it follows that

(7.2) $$\frac{\partial \log \varphi_j}{\partial y} = -\frac{\beta}{y^2}(p_j - \sum_r p_r \varphi_r), \quad \frac{\partial \log \varphi_j}{\partial \log p_j} = \frac{\beta p_j}{y}(1-\varphi_j) \quad \text{and} \quad \frac{\partial \log \varphi_j}{\partial \log p_k} = -\frac{\beta p_k \varphi_k}{y}.$$

Theorem 2 and (7.2) therefore imply the following compensated price elasticities:

$$\frac{\partial^+ \log \varphi_j^c}{\partial \log p_j} = \frac{\beta}{y} \sum_{r \neq j} p_r \varphi_r, \qquad \frac{\partial^+ \log \varphi_j^c}{\partial \log p_k} = -\frac{\beta p_j \varphi_k}{y},$$

$$\frac{\partial^- \log \varphi_j^c}{\partial \log p_j} = \frac{\partial \log \varphi_j}{\partial \log p_j} = \frac{\beta p_j}{y}(1-\varphi_j) \quad \text{and} \quad \frac{\partial^- \log \varphi_j^c}{\partial \log p_k} = \frac{\partial \log \varphi_j}{\partial \log p_k} = -\frac{\beta p_k \varphi_k}{y}$$

for $k \neq j$. From these results it follows that the income effects associated with a price increase are

(7.3) $$\frac{\partial^- \log \varphi_j^c}{\partial \log p_j} - \frac{\partial^+ \log \varphi_j^c}{\partial \log p_j} = \frac{\partial \log \varphi_j}{\partial \log p_j} - \frac{\partial^+ \log \varphi_j^c}{\partial \log p_j} = -\frac{\beta}{y} \sum_r (p_r - p_j)\varphi_r$$

and

(7.4) $$\frac{\partial^- \log \varphi_j^c}{\partial \log p_k} - \frac{\partial^+ \log \varphi_j^c}{\partial \log p_k} = \frac{\partial \log \varphi_j}{\partial \log p_k} - \frac{\partial^+ \log \varphi_j^c}{\partial \log p_k} = -\frac{\beta (p_k - p_j)\varphi_k}{y}.$$

From (7.2) and (7.3) we obtain that the conditional income effects associated with a price increase relative to the substitution effects are

(7.5) $$\frac{E(\partial \log \varphi_j(p,t,y) / \partial \log p_j \mid y)}{E(\partial^+ \log \varphi_j^c(p,t,y) / \partial \log p_j \mid y)} - 1 = -\frac{\sum_r (p_r - p_j)\bar{\varphi}_r(p,y)}{\sum_{r \neq j} p_r \bar{\varphi}_r(p,y)}$$

and

(7.6) $$\frac{\partial \log \varphi_j(p,t,y) / \partial \log p_k}{\partial^+ \log \varphi_j^c(p,t,y) / \partial \log p_k} - 1 = \frac{(p_k - p_j)}{p_j}$$

where $\bar{\varphi}_j(p,y) = E(\varphi_j(p,t,y) \mid y)$ is the aggregate choice probability conditional on income. The formula in (7.6) shows that one can express the cross income effect relative to the cross substitution effect simply by knowledge of the prices. If $Var(1/y)$ is small, it follows from (7.5) that

(7.7) $$\frac{E\partial \log \varphi_j(p,t,y) / \partial \log p_j}{E\partial^+ \log \varphi_j^c(p,t,y) / \partial \log p_j} - 1 \cong -\frac{\sum_r (p_r - p_j)\bar{\bar{\varphi}}_r(p)}{\sum_{r \neq j} p_r \bar{\bar{\varphi}}_r(p)}$$

where $\bar{\bar{\varphi}}_j(p) = E\varphi_j(p,t,y)$. The formula in (7.7) show that one can express the aggregate direct income effects relative to the aggregate direct substitution effects, approximately, solely by the prices and aggregate transportation shares, without knowing the parameter estimates.

Similarly, it follows from Corollary 5 that the compensated elasticities with respect to travel times are given by

$$\frac{\partial^+ \log \varphi_j^c}{\partial \log t_{js}} = \frac{\gamma_s t_{js}}{p_j} \sum\nolimits_{r \neq j} p_r \varphi_r, \quad \frac{\partial^+ \log \varphi_j^c}{\partial \log t_{ks}} = -\frac{\gamma_s p_j t_{ks} \varphi_k}{p_k},$$

$$\frac{\partial^- \log \varphi_j^c}{\partial \log t_{js}} = \frac{\partial \log \varphi_j}{\partial \log t_{js}} = \gamma_s t_{js}(1-\varphi_j), \quad \frac{\partial^- \log \varphi_j^c}{\partial \log t_{ks}} = \frac{\partial \log \varphi_j}{\partial \log t_{ks}} = -\gamma_s t_{ks} \varphi_k$$

with corresponding income effects associated with an increase in travelling times that become,

(7.8) $$\frac{\partial^- \log \varphi_j^c}{\partial \log t_{js}} - \frac{\partial^+ \log \varphi_j^c}{\partial \log t_{js}} = \frac{\partial \log \varphi_j}{\partial \log t_{js}} - \frac{\partial^+ \log \varphi_j^c}{\partial \log t_{js}} = -\frac{\gamma_s t_{js}}{p_j} \sum_r (p_r - p_j)\varphi_r$$

and

(7.9) $$\frac{\partial^- \log \varphi_j^c}{\partial \log t_{ks}} - \frac{\partial^+ \log \varphi_j^c}{\partial \log t_{ks}} = \frac{\partial \log \varphi_j}{\partial \log t_{ks}} - \frac{\partial^+ \log \varphi_j^c}{\partial \log t_{ks}} = -\frac{\gamma_s t_{ks}(p_k - p_j)\varphi_k}{p_k}$$

for s = 1, 2. From (7.3) to (7.9) we realize that the income effects may be substantial if the prices differ substantially. In contrast, there are no income effects when prices are equal.

Exampe 2: Urban travel demand with endogenous price of one alternative

In this example, the choice model is the same as given in (7.1), where alternative one is a public rail transit service where the price is determined as in models of oligopolistic competition (Anderson et al., 1992). Let c denote the marginal cost (assumed to be constant) of travelling for one passenger on the transit system. The management of the transit system is assumed to know the aggregate choice probability $\bar{\bar{\varphi}}_1(p)$. The expected profit therefore becomes $(p_1 - c)\bar{\bar{\varphi}}_1(p) - K$ where K represents sunk cost. Let $\mu = Ey^{-1}$. If $Var(y^{-1})$ is small we can write $\bar{\bar{\varphi}}_1(p) \cong \bar{\varphi}_1(p,\mu)$. By taking the prices of other alternatives as given, the price of a transit ride which maximizes expected profit is determined (approximately) by

(7.10) $$p_1 \cong c - \frac{1}{\beta\mu(1-\bar{\varphi}_1(p,\mu))}.$$

From (7.10) it follows by implicit differentiation that

(7.11) $$\frac{\partial p_1}{\partial c} \cong 1 - \bar{\varphi}_1(p,\mu) \cong 1 - \bar{\bar{\varphi}}_1(p).$$

Hence, (7.3), (7.4) and (7.11) imply that the income effects associated with an increase in the marginal cost c becomes

$$\frac{\partial \log \varphi_1}{\partial \log c} - \frac{\partial^+ \log \varphi_1^c}{\partial \log c} = \left(\frac{\partial \log \varphi_j}{\partial \log p_1} - \frac{\partial^+ \log \varphi_j^c}{\partial \log p_1} \right) \frac{\partial \log p_1}{\partial \log c} \cong -\frac{\beta}{y}(1-\bar{\bar{\varphi}}_1)\sum_r (p_r - p_1)\varphi_r$$

and

$$\frac{\partial \log \varphi_j}{\partial \log c} - \frac{\partial^+ \log \varphi_j^c}{\partial \log c} = \left(\frac{\partial \log \varphi_j}{\partial \log p_1} - \frac{\partial^+ \log \varphi_j^c}{\partial \log p_1} \right) \frac{\partial \log p_1}{\partial \log c} \cong -\frac{\beta(1-\bar{\bar{\varphi}}_1)(p_1 - p_j)\varphi_1}{y}.$$

Example 3: Labor force participation with non-linear taxes and fixed hours of work

Consider the following model of labor force participation of married women. The women face the choice of working full-time (alternative 2) with wage income w, or not working (alternative 1). Hours of work h is normalized to 1. Let y be non-labor income (husband's income) and $f(w,y)$ is the function that transforms gross labor income w and non-labor income y to income after taxes. We assume that $f(w,y)$ is continuously differentiable. The agent's utility in disposable income C and hours of work h is given by the separable functional form

$$U(C,h) = \frac{C^\alpha - 1}{\alpha} + g(h)$$

where $g(h)$ is a decreasing function and $\alpha < 1$ is a parameter. Let

(7.12) $$U_2 = \frac{f(w,y)^\alpha - 1}{\alpha} + b_2 + \varepsilon_2 \quad \text{and} \quad U_1 = \frac{f(0,y)^\alpha - 1}{\alpha} + b_1 + \varepsilon_1$$

It follows that the agent will work if

$$\frac{f(w,y)^\alpha - 1}{\alpha} + g(1) > \frac{f(0,y)^\alpha - 1}{\alpha} + g(0)$$

which is equivalent to $U_2 > U_1$ where $b_1 + \varepsilon_1 = g(0)$, $b_2 + \varepsilon_2 = g(1)$, ε_1 and ε_2 are zero mean stochastic taste shifters. Let F be the c.d.f. of $\varepsilon_1 - \varepsilon_2$. The probability of working becomes

(7.13) $$\varphi_2 = \varphi_2(w,y) = F\left(\frac{f(w,y)^\alpha - 1}{\alpha} - \frac{f(0,y)^\alpha - 1}{\alpha} - b \right)$$

where $b = b_1 - b_2$ When $\alpha = 0$ the probability of working is defined by

$$\varphi_2 = F\left(\log f(w,y) - \log f(0,y) - b\right).$$

Note that in this example utility is increasing in price (wage income). This implies that the right marginal compensated wage effect is equal to the corresponding marginal uncompensated wage effect whereas the left marginal compensated wage effects differ from the corresponding marginal uncompensated effects (Corollary 2).

Furthermore, it follows from (7.12), (7.13) and Corollary 2 that

$$\frac{\partial^+ \varphi_2^c}{\partial w} = \frac{\partial \varphi_2}{\partial w} = f(w,y)^{\alpha-1} f_1'(w,y) F'\left(\frac{f(w,y)^\alpha - 1}{\alpha} - \frac{f(0,y)^\alpha - 1}{\alpha} - b \right),$$

$$\frac{\partial \varphi_2}{\partial y} = \left(f(w,y)^{\alpha-1} f_2'(w,y) - f(0,y)^{\alpha-1} f_2'(0,y) \right) F'\left(\frac{f(w,y)^\alpha - 1}{\alpha} - \frac{f(0,y)^\alpha - 1}{\alpha} - b \right)$$

and

$$\frac{\partial^- \varphi_2^c}{\partial w} = \frac{\partial \varphi_2}{\partial w} - \frac{\partial v_2(w,y)/\partial w}{\partial v_2(w,y)/\partial y} \cdot \frac{\partial \varphi_2}{\partial y} = \frac{\partial \varphi_2}{\partial w} - \frac{f_1'(w,y)}{f_2'(w,y)} \cdot \frac{\partial \varphi_2}{\partial y}.$$

Thus, in this case we obtain that the income effect relative to the substitution effect of a wage decrease becomes

$$\text{(7.14)} \qquad \frac{\partial \varphi_2 / \partial w}{\partial^- \varphi_2^c / \partial w} - 1 = \frac{f_1'(w,y) \partial \varphi_2 / \partial y}{f_2'(w,y) \partial \varphi_2 / \partial w - f_1'(w,y) \partial \varphi_2 / \partial y} = \frac{f(w,y)^{\alpha-1} f_2'(w,y)}{f(0,y)^{\alpha-1} f_2'(0,y)} - 1.$$

We note that the income effect relative to the substitution effect can be expressed by a simple formula provided the Box-Cox parameter α has been estimated. When $\alpha = 1$ the formula in (7.14) reduces to

$$\text{(7.15)} \qquad \frac{\partial \log \varphi_2 / \partial \log w}{\partial^- \log \varphi_2^c / \log w} - 1 = \frac{f_2'(w,y)}{f_2'(0,y)} - 1.$$

The relation in (7.15) shows that even when utility is linear in disposable income the income effect is still different from zero if taxes are non-linear. If husband and wife are taxed separately, then $f_2'(0,y) = f_2'(w,y)$, implying that in this case the income effect vanishes.

8. Conclusions

In this paper we have discussed marginal compensated effects in discrete choice models and established an aggregate Slutsky equation (discrete Slutsky equation) for such models. We have shown that an earlier analysis of marginal compensated effects is incomplete and incorrect. The discrete Slutsky equation has the (asymmetric) property that the marginal compensated price effects in the case of a price increase may differ from the marginal compensated price effects in the case of a price decrease, Moreover, we have discussed why the discrete Slutsky equation is asymmetric. As we have demonstrated by discussing selected examples, the discrete Slutsky equation is very practical to use provided that the systematic part of the utility function has been estimated.

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Appendix

Proofs

Before we give the proofs of the theorems above we need the following lemma.

Lemma 1

Let $U_j(p_j, y) = v_j(p_j, y) + \varepsilon_j = v_j + \varepsilon_j,\ j = 1,2,...,m,$ *where the random terms* $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ *have joint c. d. f. F and deterministic terms* $\{v_j(p_j, y)\}$ *that are strictly monotone in* p_j *and strictly increasing in y. Let* $\tilde{p}$ *be the vector of ex post prices. Let* $\psi_j(z) = \max(v_j, v_j(\tilde{p}_j, z))$ *and let* y_j *be determined by* $v_j = v_j(\tilde{p}_j, y_j)$. *Then*

(i) $$P(\max_r U_r(p_r, y) = U_j(p_j, y) = \max_r U_r(\tilde{p}_r, Y_{jk}) = U_k(\tilde{p}_k, Y_{jk}) \in du, Y_{jk} \in dz)$$
$$= F''_{jk}(u - \psi_1(z), u - \psi_2(z), ..., u - \psi_m(z)) v'_{k2}(p_k, z) du dz$$

for $y_j > z \geq y_k$, $k \neq j$, *and*

(ii) $$P(\max_r U_r(p_r, y) = U_j(p_j, y) = \max_r U_r(\tilde{p}_r, Y_{jj}) \in du, Y_{jj} = y_j)$$
$$= F'_j(u - \psi_1(y_j), ..., u - v_j, ..., u - \psi_m(y_j)) du.$$

Proof of Lemma 1:

Consider first the proof of (i). Let J and $\tilde{J}$ denote the ex ante and ex post choice given that the ex-ante and ex post utility levels of the chosen alternatives are equal. For notational convenience, let $U_r = U_r(p_r, y)$ and $\tilde{U}_r(Y) = U_r(\tilde{p}_r, Y)$. For $j \neq k$ we have that

$$\{J = j, \tilde{J} = k\} \Leftrightarrow \{\max_{r \neq \{j,k\}} U_r \leq U_j, \max_{r \neq k} \tilde{U}_r(Y_{jk}) \leq U_j\}.$$

For alternative *j* to be the most preferred alternative ex ante and alternative *k* the most preferred alternative ex post, one must have $\tilde{U}_k(Y_{jk}) = U_j > U_k$, which implies that $Y_{jk} > y_k$. Furthermore, since alternative k is the most preferred one ex post, $\tilde{U}_k(Y_{jk}) > \tilde{U}_j(Y)$, which implies that $\tilde{U}_j(Y_{jk}) < U_j$ and $Y_{jk} < y_j$. Accordingly, the event $\{J = j, \tilde{J} = k\}$ has positive probability only if $y_j > y_k$. Moreover, the event $\{\tilde{U}_k(z) = U_j\}$ implies that $\{\tilde{U}_j(z) \leq U_j, U_k \leq U_j\}$. Accordingly, the relation above yields

$$\{J = j, \tilde{J} = k, Y_{jk} = z\} \Leftrightarrow \{\max_{r \neq \{j,k\}} \max(U_r, \tilde{U}_r(z)) \leq U_j = \tilde{U}_k(z)\}.$$

Thus, the corresponding probabilities are therefore given by

$$P(J = j, \tilde{J} = k, Y_{jk} \in [z, z+\Delta z), U_j \in (u, u+\Delta u))$$

$$= P(\max_{r\neq j} U_r \leq u, \max_{r\neq k} \tilde{U}_r(Y_{jk}) \leq \tilde{U}_k(Y_{jk}), Y_{jk} \in (z, z+\Delta z), U_j \in (u, u+\Delta u)) + o(\Delta u \Delta z)$$

$$= P(\max_{r\neq j} U_r \leq u, \max_{r\neq k} \tilde{U}_r(z) \leq u, \tilde{U}_k(z) \leq u < \tilde{U}_k(z+\Delta z), U_j \in (u, u+\Delta u)) + o(\Delta u \Delta z)$$

$$= P(\max_{r\neq \{j,k\}} \max(U_r, \tilde{U}_r(z)) \leq u, \tilde{U}_l(z) \leq u < \tilde{U}_k(z+\Delta z), U_j \in (u, u+\Delta u)) + o(\Delta u \Delta z)$$

$$= P(\max_{r\neq \{j,k\}} (\psi_r(z)+\varepsilon_r) \leq u, \tilde{U}_k(z) \leq u < \tilde{U}_k(z+\Delta z), U_j \in (u, u+\Delta u)) + o(\Delta u \Delta z)$$

$$= F_j'(u-\psi_1(z), u-\psi_2(z), ..., u-\psi_m(z))\Delta u + o(\Delta u \Delta z)$$

$$-F_1'(u-\psi_1(z), ..., u-\psi_k(z+\Delta z), ..., u-\psi_m(z))\Delta u$$

$$= F_{jk}''(u-\psi_1(z), u-\psi_2(z), ..., u-\psi_m(z)) v_{k2}'(p_k, z)\Delta u \Delta z + o(\Delta z \Delta u).$$

This proves (*i*) of Lemma A1. Consider now the second part (*ii*). We have that

$$P(\max_{r\neq j} U_r \leq U_j \geq \max_{r\neq j} \tilde{U}_r(Y_{jj}) \in (u, u+\Delta u), Y_{jj} = y_j)$$

$$= P(\max_{r\neq j} U_r \leq U_j \geq \max_{r\neq j} \tilde{U}_r(y_j) \in (u, u+\Delta u))$$

$$= P(\max_{r\neq j} (\varepsilon_r + \psi_r(y_j)) \leq v_j + \varepsilon_j \in (u, u+\Delta u))$$

$$= F_j'(u-\psi_1(y_j), ..., u-v_j, ..., u-\psi_m(y_j))\Delta u + o(\Delta u),$$

which proves the second part.

Q. E. D.

Proof of Theorem 1:

Recall that

$$H_j'(v_1, v_2, ..., v_m) \equiv P(v_j + \varepsilon_j = \max_r (v_r + \varepsilon_r)) = \int_{-\infty}^{\infty} F_j'(u-v_1, u-v_2, .., u-v_m) du.$$

From (*i*) in Lemma 1 (with the same notation as in Lemma 1) it follows that

(A.1) $$P(J = j, \tilde{J} = k, Y_{jk} \in [z, z+\Delta z))$$

$$= v_{k2}'(\tilde{p}_k, z)\Delta z \int_{-\infty}^{\infty} F_{jk}''(u-\psi_1(z), u-v_k(\tilde{p}_k, z), ..., u-\psi_m(z)) du$$

$$= -v_{k2}'(\tilde{p}_k, z)\Delta z H_{jk}''(\psi_1(z), v_k(\tilde{p}_k, z), ..., \psi_m(z)) + o(\Delta z).$$

Evidently, differentiation under the integral above is allowed in this case. Furthermore, since alternative j is chosen ex ante and alternative k ex post it must be the case that $U_j(p_j, y) > U_j(\tilde{p}_j, Y_{jk})$ and $U_k(\tilde{p}_k, Y_{jk}) > U_k(p_k, y)$ implying that $v_j(p_j, y) > v_j(\tilde{p}_j, Y_{jk})$ and $v_k(\tilde{p}_k, Y_{jk}) > v_k(p_k, y)$. Hence, it must be true that the probability in (A.1) vanished unless

$y_k \le z \le y_j$. By integrating (A.1) with respect to z between y_k and y_j yields (4.3). The relation in (4.4) follows from Lemma 1 (ii).

Q. E. D.

Proof of Theorem 2:

Recall that $v_r(w_r, y)$ is strictly decreasing in prices and strictly increasing in income. Assume first that the price of alternative *j* increases from p_j to $\tilde{p}_j = p_j + \Delta p_j$ where Δp_j is small and positive. Then, $y_j(\tilde{p}_j) > y$ and $y_r = y$ for $r \ne j$. Hence, it follows from Lemma 1 that $Q^c(r, j) = 0$ for $r \ne j$. Furthermore, from the definition of $y_j(\tilde{p}_j)$ it follows that $v_j(\tilde{p}_j, y_j(\tilde{p}_j)) = v_j(p_j, y)$. Implicit differentiation of the latter equation with respect to *u* yields

(A.2)
$$\frac{\partial y_j(p_j)}{\partial p_j} = y_j' = -\frac{v_{j1}'(p_j, y)}{v_{j2}'(p_j, y)} = -\frac{v_{j1}'}{v_{j2}'}.$$

Since $y_j > y$, $\max_r(v_r(p_r, y), v_r(p_r, y_j)) = v_r(p_r, y_j)$ and we get from Lemma 1, (2.4) and (A.2) that

(A.3)
$$\sum_r Q^c(r, j) - \varphi_j = Q^c(j, j) - \varphi_j$$

$$= H_j'(v_1(p_1, y_j), v_2(p_2, y_j), ..., v_j(p_j, y), ..., v_m(p_m, y_j)) - H_j'(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y))$$

$$= y_j' \sum_{r \ne j} H_{jr}''(v_1(p_1, y), v_2(p_2, y), ..., v_j(p_j, y), ..., v_m(p_m, y)) v_{r2}'(p_r, y) \Delta p_j + o(\Delta p_j)$$

$$= \frac{y_j' \partial H_j'(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y)) \Delta p_j}{\partial y}$$

$$- y_j' H_{jj}''(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y)) v_{j2}'(p_j, y) \Delta p_j + o(\Delta p_j)$$

$$= \frac{\partial H_j'(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y)) y_j' \Delta p_j}{\partial y} + \frac{\partial H_j'(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y)) \Delta p_j}{\partial p_j}$$

$$+ o(\Delta p_j) = \left(\frac{\partial \varphi_j}{\partial p_j} + y_j' \cdot \frac{\partial \varphi_j}{\partial y} \right) \Delta p_j + o(\Delta p_j) = \left(\frac{\partial \varphi_j}{\partial p_j} - \frac{v_{j1}'}{v_{j2}'} \cdot \frac{\partial \varphi_j}{\partial y} \right) \Delta p_j + o(\Delta p_j)$$

$$= \left(\frac{\partial \varphi_j}{\partial p_j} + x_j \cdot \frac{\partial \varphi_j}{\partial y} \right) \Delta p_j + o(\Delta p_j).$$

Since

$$\frac{\partial^+ \varphi_j^c}{\partial p_j} = \lim_{\Delta p_j \to 0} \left(\frac{\sum_r Q^c(r, j) - \varphi_j}{\Delta p_j} \right)$$

the first part of the theorem follows from (A.3).

Consider next the corresponding cross price effects. That is, we consider the marginal compensated effect on the choice of alternative *j* when the price of alternative *k* increases where $k \neq j.$ We have that $Q^c(r,j)=0$ for $r \neq k$ and $Q^c(k,j)>0.$ From Theorem 1 we obtain that

(A.4) $$Q^c(k,j) = -\int_y^{y_k(\tilde{p}_k)} H''_{kj}(\psi_1(z),\psi_2(z),..,\psi_m(z))v'_{j2}(p_k,z)dz$$

which together with (A.2) and the Mean value theorem imply that

(A.5) $$\begin{aligned} Q^c(k,j) &= -H''_{kj}(v_1(p_1,y),...,v_m(p_m,y))v'_{j2}(p_j,y)(y_k(\tilde{p}_k)-y)+o(\Delta p_k) \\ &= -\frac{\partial H'_j(v_1(p_1,y),...,v_m(p_m,y))v'_{j2}(p_j,y)(y_k(\tilde{p}_k)-y)}{v'_{k1}(p_k,y)\partial p_k}+o(\Delta p_k) \\ &= -\frac{\partial \varphi_j}{\partial p_k}\cdot\frac{v'_{j2}(p_k,y)}{v'_{k1}(p_k,y)}\cdot\frac{\partial y_k(p_k)\Delta p_k}{\partial p_k}+o(\Delta p_k) = \frac{\partial \varphi_j}{\partial p_k}\cdot\frac{v'_{j2}(p_j,y)\Delta p_k}{v'_{k2}(p_k,y)}+o(\Delta p_k). \end{aligned}$$

Since $y_j = y$ and $\psi_k(y_j) = \max(v_k(\tilde{p}_k,y),v_k(p_k,y)) = v_k(p_k,y),$ it follows from Theorem 1 that

(A.6) $$Q^c(j,j) = \varphi_j.$$

From (A.4) and (A.5) it thus follows that

$$\frac{\partial^+\varphi_j^c}{\partial p_k} = \lim_{\Delta p_k \to 0}\left(\frac{Q^c(k,j)+Q^c(j,j)-\varphi_j}{\Delta p_k}\right) = \lim_{\Delta p_k \to 0}\frac{Q^c(k,j)}{\Delta p_k} = \frac{\partial \varphi_j}{\partial p_k}\cdot\frac{v'_{j2}(p_j,y)}{v'_{k2}(p_k,y)}.$$

Consider next the marginal compensated own price effect in the case when $\Delta p_j < 0.$ Then it follows that $y_j < y$ so that $\psi_r(y_j) = v_r(p_r,y)$ for all *r*, implying that $Q^c(j,j)=\varphi_j$ according to Theorem 1. Furthermore, from Theorem 1 and the Mean value theorem we get, for some $z^* \in (y_j,y)$ that

$$\begin{aligned} Q^c(r,j) &= -H''_{rj}(\psi_1(z^*),\psi_2(z^*),...,\psi_m(z^*))v'_{j2}(\tilde{p}_j,z^*)(y-y_j(\tilde{p}_j))+o(\Delta p_j) \\ &= -H''_{rj}(v_1(p_1,y),v_2(p_2,y),...,v_2(p_2,y))v'_{j2}(p_j,y)(y-y_j(\tilde{p}_j))+o(\Delta p_j) \\ &= H''_{rj}(v_1(p_1,y),v_2(p_2,y),...,v_m(p_m,y))v'_{j2}(p_j,y)\Delta p_j y'_j(p_j)+o(\Delta p_j) \\ &= -H''_{rj}(v_1(p_1,y),v_2(p_2,y),...,v_m(p_m,y))\frac{v'_{j2}(p_j,y)\Delta p_j v'_{j1}(p_j,y)}{v'_{j2}(p_j,y)}+o(\Delta p_j) \\ &= -H''_{rj}(v_1(p_1,x),v_2(p_2,y),...,v_m(p_m,y))\Delta p_j v'_{j1}(\tilde{p}_j,y)+o(\Delta p_j). \end{aligned}$$

Consequently, we obtain that

$$\sum_r Q^c(r,j)-\varphi_j = \sum_{r\neq j} Q^c(r,j)$$

$$= -\sum_{r \neq j} H''_{rj}(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y))\Delta p_j v'_{j1}(p_j, y) + o(\Delta p_j)$$

$$= -\sum_{k \neq j} \frac{\partial \varphi_k}{\partial p_j} \cdot \Delta p_j + o(\Delta p_j) = \frac{\partial \varphi_j}{\partial p_j} \Delta p_j + o(\Delta p_j)$$

which implies that

$$\frac{\partial^- \varphi_j^c}{\partial p_j} = \frac{\partial \varphi_j}{\partial p_j}.$$

Finally, consider the marginal compensated cross price effect when $\Delta p_k < 0.$ In this case we get from Theorem 1 that $Q^c(k, j) = 0$ when $k \neq j,$ and

$$Q^c(j, j) = H'_j(v_1(p_1, y), ..., v_{k-1}(p_{k-1}, y), v_k(\tilde{p}_k, y), v_{k+1}(p_{k+1}, y), ..., v_m(p_m, y)).$$

because $y_j = y.$ Therefore, we get that

$$\sum_r Q^c(r, j) - \varphi_j = Q^c(j, j) - \varphi_j$$

$$= H'_j(v_1(p_1, y), ..., v_j(\tilde{p}_k, y), ..., v_m(p_m, y)) - \varphi_j.$$

By first order Taylor expansion the last expression becomes

$$H''_{jk}(v_1(p_1, y), v_2(p_2, y), ..., v_m(p_m, y))v'_{k1}(p_k, y)\Delta p_k + o(\Delta p_j) = \frac{\partial \varphi_j \Delta p_k}{\partial p_k} + o(\Delta p_k)$$

which implies that

$$\frac{\partial^- \varphi_j^c}{\partial p_k} = \frac{\partial \varphi_j}{\partial p_k}.$$

Q. E. D.