Aaberge, R. (2007): " Gini's nuclear family." Journal of Economic Inequality, **vol. 5,** no. 3, 305-322. Springer.

Title: Gini's nuclear family

Author: Aaberge, Rolf

Version: **Authors final version / Peer reviewed (Post-print)**

Note: The original publication is available at www.springer.com

Publisher: Springer

DOI: http://dx.doi.org/10.1007/s10888-006-9050-8

Gini's Nuclear Family

Rolf Aaberge

Research Department

Statistics Norway

(*Journal of Economic Inequality*, **5**, 305 - 322, 2007)

Abstract

The purpose of this paper is to justify the use of the Gini coefficient and two close relatives for summarizing the basic information of inequality in distributions of income. To this end we employ a specific transformation of the Lorenz curve, the scaled conditional mean curve, rather than the Lorenz curve as the basic formal representation of inequality in distributions of income. The scaled conditional mean curve is shown to possess several attractive properties as an alternative interpretation of the information content of the Lorenz curve and furthermore proves to yield essential information on polarization in the population. The paper also provides asymptotic distribution results for the empirical scaled conditional mean curve and the related family of empirical measures of inequality.

Keywords: The scaled conditional mean curve, measures of inequality, the Gini coefficient, the Bonferroni coefficient, measures of social welfare, principles of transfer sensitivity, estimation, asymptotic distributions.

JEL classification: D3, D63.

Acknowledgement: This paper was written when the author was visiting ICER in Torino. ICER and the Norwegian Research Council are gratefully acknowledged for providing financial support. I wish to thank Jacques Silber, Terje Skjerpen and two referees for helpful comments.

Address: Research Department, Statistics Norway, P.O.B. 8131 Dep., N-0033 Oslo, Norway
E-mail: rolf.aaberge@ssb.no

1. Introduction

Empirical analyses of inequality in income distributions are conventionally based on the Lorenz curve. To summarize the information content of the Lorenz curve and to achieve rankings of intersecting Lorenz curves the standard approach is to employ the Gini coefficient in combination with one or two inequality measures from the Atkinson family or the Theil family. However, since the Gini coefficient and Atkinson's and Theil's measures of inequality have distinct theoretical foundations it is difficult to evaluate their capacity as complementary measures of inequality[1].

By exploiting the fact that the Lorenz curve can be considered analogous to a cumulative distribution function, Aaberge (2000) draws on standard statistical practice to justify the use of the first few moments of the Lorenz curve (LC-moments) as basis for summarizing the information content of the Lorenz curve. However, considered as a group these measures suffer from a drawback since none of them in general are particularly sensitive to changes that concern the lower part of the income distribution. The reason why the moments of the Lorenz curve in most cases are more sensitive to changes that take place in the central and upper part rather than in the lower part of the income distribution is simply due to the fact that the Lorenz curve has a convex functional form. Thus, even though the first three LC-moments in many cases jointly provide a good description of the inequality in an income distribution it would for informational reasons as well as for the sake of interpretation be preferable to employ a few measures of inequality that also prove to supplement each other with regard to sensitivity to transfers at the lower, the central and the upper part of the income distribution. To this end Section 2 provides arguments for using a specific transformation of the Lorenz curve, the scaled conditional mean curve, rather than the Lorenz curve as basis for introducing and justifying application of a few measures for summarizing inequality in income distributions. The scaled conditional mean curve turns out to possess several useful properties which will be discussed below. Section 3 demonstrates that the moments of the scaled conditional mean curve define a convenient family of inequality measures where the first three moments prove to supplement each other with regard to focus on the lower, the central and the upper part of the income distribution. Section 4 deals with estimation and asymptotic distribution theory for the empirical scaled conditional mean curve and the related family of empirical measures of inequality. Moreover, an empirical illustration based on Norwegian data for 1986-1998 is also provided. Section 5 summarizes the paper.

2. The scaled conditional mean curve

Let X be an income variable with cumulative distribution function $F(\cdot)$, density $f(\cdot)$ and mean μ. Let $[0,\infty>$ be the domain of F where $F^{-1}(0) \equiv 0$. The Lorenz curve $L(\cdot)$ for F is defined by

---

[1] See Giorgi (1990) for a bibliographic portrait of the Gini coefficient.

(1) $$L(u)=\frac{1}{\mu}\int_0^u F^{-1}(t)\,dt, \quad 0\le u\le 1,$$

where F^{-1} is the left inverse of F. Note that F can either be a discrete or a continuous distribution function. Although the former is what we actually observe, the latter often allows simpler derivation of theoretical results and is a valid large sample approximation. Thus, in most cases below F will be assumed to be a continuous distribution function.

The Lorenz curve is concerned with shares of income rather than relative levels of income and differs in that respect from the decile-specific presentation of income inequality which displays decile-specific mean incomes as fractions of the overall mean income. This method of presentation is frequently used by national bureaus of statistics and by researchers dealing with analyzing distributions of income. By introducing a simple transformation of the Lorenz curve we obtain an alternative interpretation of the information content of the Lorenz curve which proves to be closely related to the conventional decile-specific approach mentioned above. To this end we use the scaled conditional mean curve M(·) introduced by Aaberge (1982) and defined by[2]

(2) $$M(u)\equiv\frac{E\left[X|X\le F^{-1}(u)\right]}{\mu}=\begin{cases}\frac{1}{u\mu}\int_0^u F^{-1}(t)\,dt, & 0<u\le 1\\ 0\ , & u=0.\end{cases}$$

When inserting for (1) in (2) the following simple relationship between the scaled conditional mean curve and the Lorenz curve emerges,

(3) $$M(u)=\begin{cases}\frac{L(u)}{u}, & 0<u\le 1\\ 0\ , & u=0,\end{cases}$$

where $M(1)=1$ and $\lim_{u\to 0}\left(L(u)/u\right)=M(0)$. Thus, formally the scaled conditional mean curve is a representation of inequality that is equivalent to the Lorenz curve.

The scaled conditional mean curve possesses several attractive properties. First, it provides a convenient alternative interpretation of the information content of the Lorenz curve. For a fixed u, M(u) is the ratio between the mean income of the poorest 100u per cent of the population and the overall mean. Thus, the scaled conditional mean curve may also yield essential information on poverty, provided that we know the poverty line. The egalitarian reference line of M coincides with the horizontal line joining the points (0,1) and (1,1). At the other extreme, when one person earns the

[2] This ratio was also considered by Nygård and Sandström (1981), but they did not explore its properties as a function that is uniquely determined by the Lorenz curve, whilst Atkinson and Bourguignon (1989) used the numerator as an alternative interpretation of the information provided by the generalized Lorenz curve.

whole income, the scaled conditional mean curve coincides with the horizontal axis except for $u = 1$. Second, the scaled conditional mean curve of a uniform (0,a) distribution proves to be the diagonal line joining the points (0,0) and (1,1) and thus represents a useful additional reference line. Thus, when a M-curve intersects the diagonal line once from below (single intersection) the corresponding distribution exhibits lower inequality than a uniform (0, a) distribution below the intersection point and higher inequality than a uniform (0, a) distribution above the intersection point. Note that incomes are uniformly distributed over (0, a) if any income in this interval occurs equally frequent. Third, the family of scaled conditional mean curves is bounded by the unit square. Therefore visually, there is a sharper distinction between two different scaled conditional mean curves than between the two corresponding Lorenz curves. This distinction appears to be particular visible at the lower parts of the income distributions[3]. As an illustration Figures 1 and 2 give the Lorenz curves and the scaled conditional mean curves of the distributions of average annual earnings in Norway for the periods 1981-1982 and 1986-1987.

[Insert Figures 1 and 2 here]

As can be seen from the scaled conditional mean curves there may be differences in inequality between the lower tails of two distribution functions which may be perceived as negligible when the judgment relies on the plots of the corresponding Lorenz curves. Note, however, that a judgment of the statistical significance of this difference in inequality does not depend on whether we rely on the scaled conditional mean curve or the Lorenz curve. However, the question of whether a difference or change in inequality is large or small is separate from that of statistical significance, and appears to be more easy to deal with when we rely on plots of the scaled conditional mean curve rather than on plots of the Lorenz curve[4].

In contrast to the Lorenz curve, which always is a convex function, the shape of the scaled conditional mean curve proves to be strongly related to the shape of the underlying distribution function. In order to demonstrate this fact observe that the first derivative of M is non-negative and that the second derivative of M is given by

(4) $$M''(u) = -\frac{1}{\mu u^3}\int_0^u \frac{t^2 f'\left(F^{-1}(t)\right)}{f^3\left(F^{-1}(t)\right)}dt,$$

---

[3] Atkinson and Bourguignon (1989) brought forward this property to justify the use of the "incomplete mean curve" (the numerator of M) rather than the generalized Lorenz curve.

[4] The estimates of Figures 1 and 2 are based on data relative to 621 804 persons available from Statistics Norway's Tax Assessment Files. Thus, sampling errors are of minor importance in this case.

provided that $\left[u^2 / f\left(F^{-1}(u)\right)\right] \to 0$ when $u \to 0+$. The expression (4) for the second derivative of M demonstrates that there is a close relationship between the shape of the distribution function F and the shape of the scaled conditional mean curve. For example, when F is convex, i.e. F is strongly skew to the left, then M is concave. In this case, a minority of the population is poor and the majority is rich. By contrast, when F is concave, i.e. F is strongly skew to the right, then M is convex. In this case the majority of the population has low incomes, whereas a minority has high incomes. Moreover, a symmetric and convex/concave distribution function F implies a concave/convex shape of the corresponding scaled conditional mean curve, whereas a symmetric and concave/convex F implies a convex/concave scaled conditional mean curve. Note that a concave/convex distribution function occurs when there is a tendency of polarization in the population[5]. At the extreme the concave/convex (and symmetric) F becomes a two-point distribution function, which displays complete polarization.

Under the restriction of equal mean incomes the problem of ranking scaled conditional mean curves (M-curves) or Lorenz curves formally corresponds to the problem of choosing between uncertain prospects. This relationship has been utilized by e.g. Kolm (1969) and Atkinson (1970) to characterize the criterion of non-intersecting Lorenz curves in the case of distributions with equal mean incomes. This was motivated by the fact that in cases of equal mean incomes the criterion of non-intersecting Lorenz curves is equivalent to second-degree stochastic dominance[6], which means that the criterion of non-intersecting Lorenz curves obeys the Pigou-Dalton principle of transfers. *The Pigou-Dalton principle of transfers* states that an income transfer from a richer to a poorer individual reduces income inequality, provided that their ranks in the income distribution are unchanged, and is defined formally by[7]

DEFINITION 1. (The Pigou-Dalton principle of transfers.) Consider a discrete income distribution F. A transfer $\delta > 0$ from a person with income $F^{-1}(t)$ to a person with income $F^{-1}(s)$, where the transfer is assumed to be rank-preserving, is said to reduce inequality in F when $s < t$ and raise inequality in F when $s > t$.

The following result demonstrates that the scaled conditional mean curve $M(\cdot)$ obeys *the Pigou-Dalton principle of transfers*, which means that the criterion of non-intersecting M-curves is equivalent to second-degree stochastic dominance of the corresponding cumulative distribution functions, provided that the means are equal.

---

[5] For recent discussions on polarization we refer to Esteban and Ray (1994) and Wolfson (1994).

[6] For a proof see Hardy, Littlewood and Polya (1934).
[7] Note that this definition of the Pigou-Dalton principle of transfers was proposed by Fields and Fei (1978).

THEOREM 1. Let M_1 and M_2 be members of the family of M-curves. Then the following statements are equivalent,

(i) M_1 first-degree dominates M_2

(ii) M_1 can be obtained from M_2 by a sequence of Pigou-Dalton transfers

We refer to Fields and Fei (1978) for a proof of the equivalence between (i) and (ii)[8] when the scaled conditional mean curve (M) in *Theorem 1* is replaced by the Lorenz curve. However, since M(u)=L(u)/u, the proof is also valid when the dominance condition is expressed in terms of the scaled conditional mean curve.

3. Gini's nuclear family of inequality measures

By observing that the Lorenz curve can be considered analogous to a cumulative distribution function Aaberge (2000) demonstrated that the moments of the Lorenz curve generate the following family of inequality measures

(5) $$D_k(F) = \frac{1}{k}\left((k+1)\int_0^1 u^k d\,L(u) - 1 \right), \quad k = 1, 2, ...,$$

called the Lorenz family of inequality measures[9], and moreover proved that it is strongly related to a subfamily of the extended Gini Family discussed by Donaldson and Weymark (1980, 1983) and Yitzhaki (1983). Alternatively, the members of the Lorenz family may be expressed in terms of the distribution function F in the following way,

(6) $$D_k(F) = \frac{1}{k\mu}\int F(x)\left(1 - F^k(x)\right)dx, \quad k = 1, 2, ...$$

Since the Lorenz curve is uniquely determined by its moments we can, without loss of information, restrict the examination of inequality in an income distribution F to the Lorenz family of inequality measures. However, even though we have obtained to reduce the size of the family of inequality measures from the standard infinite non-countable set to a countable set it still contains infinite members. For practical reasons it would be preferable to rely on a few measures of inequality in empirical applications. By drawing on standard statistical practice Aaberge (2000) proposed to use the first few moments of the Lorenz curve as primary quantities for measuring inequality, i.e. D_1, D_2 and

[8] See Rothschild and Stiglitz (1973) for a proof of the equivalence between (i) and (ii) in the case where the rank-preserving condition is abandoned in the definition of the Pigou-Dalton principle of transfers.

D_3, where D_1 is the Gini coefficient. These three measures may jointly give a good summarization of the information provided by the Lorenz curve but suffer from the inconvenience of generally turning their attention to changes that occur in the central and/or the upper part of the income distribution. However, a measure of inequality that primarily focuses attention on the lower tail can be obtained by introducing an appropriate linear combination of D_1, D_2 and D_3. As will be demonstrated below an alternative and more attractive strategy is to use the first three moments of the scaled conditional mean curve as primary quantities for measuring inequality in income distributions. The k^{th} order moment of the scaled conditional mean curve for income distribution F, $C_k(F)$, is defined by

(7) $$C_k(F) = \int_0^1 u^k \, d\, M(u).$$

By recalling the properties of M we immediately realize from (7) that the moments of the scaled conditional mean curve $\{C_k : k = 1,2,\ldots\}$ constitute a family of inequality measures with range [0,1]. Thus, without loss of generalization we can restrict the examination of the inequality in F to the moments of the scaled conditional mean curve. The following alternative expression of C_k,

(8) $$C_k(F) = k\int_0^1 u^{k-1}\left(1 - M(u)\right) du, \quad k = 1,2,\ldots$$

demonstrates that C_k for k>1 is adding up weighted differences between the scaled conditional mean curve and its egalitarian line. The mean (C_1) of M is equal to the area between the scaled conditional mean curve and its egalitarian line[10], the horizontal line joining the points (0,1) and (1,1) of Figure 2. The inequality measure C_1 appears to be identical to a measure of inequality that was introduced by Bonferroni (1930) as an alternative to the Gini coefficient, but since then it has for some reason been paid little attention in the economic literature[11]. By inserting for (2) in (8) when k=1 we obtain the following alternative expression for C_1,

(9) $$C_1(F) = -\frac{1}{\mu}\int F(x) \log F(x)\, dx.$$

[9] Note that this is a subfamily of a family of inequality measures that was introduced by Mehran (1976).

[10] Note that Eltetö and Frigyes (1968) proposed $M(F(\mu))$ as a measure of inequality. However, this measure is unaffected by transfers between individuals on the same side of the mean, which means that it does not satisfy the Pigou-Dalton transfer principle.

[11] For a few exceptions we refer to D'Addario (1936), Nygård and Sandström (1981), Aaberge (1982, 2000), Giorgi (1984, 1998), Chakravarty and Muliere (2003) and Aaberge, Colombino and Strøm (2004). In the latter paper the Bonferroni coefficient defines a measure of social welfare that is used for evaluating the performance of various tax systems.

Now, inserting (2) into (8) when $k = 2$ we find that the second order moment of the scaled conditional mean curve is equal to the Gini coefficient (C_2), whilst an alternative expression of the third order moment of the scaled conditional mean curve is given by (6) for k=2.

Note that $C_{k+1} = D_k$ for $k = 1, 2, ...,$ which means that the family $\{C_k : k = 1, 2, ...\}$ simply is the Lorenz family of inequality measures extended with the Bonferroni coefficient C_1. This also means that C_1 is uniquely determined by the Lorenz family measures of inequality. The explicit relationship is found by inserting for (2) in (8) when k=1 and by using Taylor-expansion for the term 1/u. Finally, inserting for (5) in the attained expression yields

(10) $$C_1(F) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{1}{k+1} D_k(F).$$

Since C_1, C_2 and C_3 represent the first, the second and the third order moments of the scaled conditional mean curve, they jointly may make up a fairly good summarization of the scaled conditional mean curve as well as of the Lorenz curve[12]. Moreover, as will be demonstrated below C_1 and C_3 complement the information provided by the Gini coefficient by turning particular attention to changes that take place in the lower and upper part of the income distribution. Due to these features of C_1, C_2 and C_3 we will treat them as a group and call them Gini's Nuclear Family of inequality measures. Thus, Gini's Nuclear Family of inequality measures can be considered as an adjustment of the group of measures (D_1, D_2 and D_3) discussed by Aaberge (2000) where C_2=D_1, C_3=D_2 and D_3 is replaced by the Bonferroni coefficient C_1.

Aaberge (2000) demonstrated that the Lorenz family of inequality measures as well as the Bonferroni coefficient can be given explicit expressions in terms of social welfare and moreover are members of the "illfare-ranked single-series Ginis" introduced by Donaldson and Weymark (1980) and discussed by Bossert (1990)[13]. The welfare function that corresponds to C_k is given by

(11) $$W_k(F) = \mu(1 - C_k) = \int_0^1 p_k(t) F^{-1}(t) dt, \quad k = 1, 2, ...,$$

where $p_k(t)$ is a weight function defined by

---

[12] Aaberge (2000) demonstrates that C_2 and C_3 also provide essential information on the shape of the income distribution.

[13] See also Sen (1974), Yitzhaki (1979), Weymark (1981), Hey and Lambert (1980), Donaldson and Weymark (1983), Yaari (1987, 1988), Ben Porath and Gilboa (1994) and Aaberge (2001) who have provided alternative characterizations of the Gini coefficient and generalized Gini measures of inequality.

(12) $$p_k(t)=\begin{cases}-\log t, & k=1\\ \frac{k}{k-1}\left(1-t^{k-1}\right), & k=2,3,....\end{cases}$$

The latter term of equation (11) follows by inserting for (2) in (8) and for (8) in the second term of (11), and then using integration by parts. Equation (11) shows that the welfare function W_k is a weighed sum of the ordered incomes, where the inequality aversion exhibited by W_k and the corresponding weight function decreases with increasing k. As $k\to\infty$, W_k approaches inequality neutrality and coincides with the linear additive welfare function defined by

(13) $$W_\infty=\int_0^1 F^{-1}(t)dt=\mu .$$

It follows by straightforward calculations that $W_k\le\mu$ for all k, and that W_k is equal to the mean μ for finite k if and only if F is the egalitarian distribution. Thus, W_k can be interpreted as the equally distributed equivalent income. As a contribution to the interpretation of the inequality aversion profiles exhibited by W_1, W_2 and W_3 (and C_1, C_2 and C_3) Table 1 provides ratios of the corresponding weights – as defined by (12) – of the median individual and the 5 per cent poorest, the 30 per cent poorest and the 5 per cent richest individual.

Table 1. Distributional weight profiles of W_1, W_2 and W_3 (and C_1, C_2 and C_3)

	W_1 (C_1) (Bonferroni)	W_2 (C_2) (Gini)	W_3 (C_3)
p(.05)/p(.5)	4.32	1.90	1.33
p(.30)/p(.5)	1.74	1.40	1.21
p(.95)/p(.5)	0.07	0.10	0.13

As suggested by Table 1 W_1 (C_1) is more sensitive than W_2 (C_2) to changes in the income distribution that concern the poor, whereas W_2 (C_2) is more sensitive than W_3 (C_3) to changes that occur in the lower part of the income distribution. For example, the weights in Table 1 demonstrate that the social weight of an additional Euro to a person located at the 5 per cent decile is 4.3 times the weight of the median income earner when C_1 (W_1) is used as a measure of inequality (social welfare), whereas it is only 1.3 times the weight of the median earner when C_3 (W_3) is used as a measure of inequality (social welfare). As is suggested by Table 1 and is easily verified from equation (8), C_1, C_2 and C_3 preserve first-degree M-curve and Lorenz dominance and thus satisfy the Pigou-Dalton principle of transfers. However, to deal with situations where M-curves or Lorenz curves intersect a more demanding principle than the Pigou-Dalton transfer principle is required. An obvious idea is to introduce a principle that places more emphasis on a given transfer the lower it occurs in the income distribution. Kolm (1976) and Mehran (1976) proposed two alternative versions of such a principle; the *principle of diminishing transfers* which requires the income difference between receivers and donors to be fixed and *the principle of positional transfer sensitivity* which requires a fixed difference in ranks between receivers and donors[14]. To provide a formal definition of the principle of diminishing transfers let I be an inequality measure and let $\Delta I_x(\delta, z)$ denote the change in I resulting from a transfer δ from a person with income x+z to a person with income x. Thus, $\Delta I_x(\delta, z)$ is a negative number[15]. Furthermore, let $\Delta I_{x,y}(\delta, z)$ be defined by

$$\Delta I_{x,y}(\delta,h) = \Delta I_y(\delta,h) - \Delta I_x(\delta,h). \tag{14}$$

Thus, $\Delta I_{x,y}(\delta, z)$ captures the difference between the effect on I resulting from a transfer δ from a person with income x+z to a person with income x and the effect from a transfer from a person with income y+z to a person with income *y*, where x<y.

DEFINITION 2A. Consider an income distribution F and a transfer δ from individuals with incomes x+z and y+z to individuals with incomes x and y, respectively, where the receivers are assumed to not become richer than the donors. Then the inequality measure J is said to satisfy *the principle of diminishing transfers* if

$$\Delta I_{x,y}(\delta, z) > 0 \text{ when } x < y. \quad .$$

Similarly, to provide a formal definition of *the principle of positional transfer sensitivity*, let J be an inequality measure and let $\Delta J_t(\delta, z)$ denote the change in J resulting from a transfer δ from a person with income $F^{-1}(t+h)$ to a person with income $F^{-1}(t)$ that leaves their ranks in the income distribution F unchanged. Thus, $\Delta J_t(\delta, z)$ is a negative number. Furthermore, let $\Delta J_{s,t}(\delta, z)$ be defined by

$$\Delta J_{s,t}(\delta,h) = \Delta J_t(\delta,h) - \Delta J_s(\delta,h). \tag{15}$$

14 We refer to Mehran (1976), Zoli (1999) and Aaberge (2004) for a discussion of the principle of positional transfer sensitivity.

15 For convenience the dependence of I on F is suppressed in the notation for I.

DEFINITION 2B. Consider an income distribution F and a rank-preserving transfer δ from individuals with ranks $s+h$ and $t+h$ to individuals with ranks s and t, respectively. Then the inequality measure J is said to satisfy *the principle of positional transfer sensitivity* if

$$\Delta J_{s,t}(\delta, z) > 0 \text{ when } s < t.$$

By applying Theorem 2 in Aaberge (2000) we find that the Bonferroni coefficient satisfy *the principle of diminishing transfers* for distribution functions that are strictly logconcave[16]. This class includes the uniform, the exponential, the Pareto, the Gamma, the Laplace, the Weibull and the Wishart distributions. For logconcave distribution functions there are, as were also noted by Heckman and Honoré (1990) and Caplin and Nalebuff (1991), a rising gap between the income of the richest and the average income of those units with income lower than the richest as we move up the income distribution[17], i.e. $x - E(Y \mid Y \le x)$ is an increasing function of x. Observe that if X and Y are distributed according to F (with mean μ) we have

$$C_1(F) = \frac{E\left\{X - E\left(Y \mid Y \le X\right)\right\}}{\mu} \tag{16}$$

which means that the Bonferroni coefficient is equal to the ratio between the mean of these income gaps and the overall mean income. Consequently, the Bonferroni coefficient assigns more weight to transfers taking place lower down in the distribution for all distributions which are strongly skewed to the right and even for some distributions which are strongly skewed to the left. Distributions which are strongly skewed to the left exhibit a minority of poor individuals/households and a majority of rich individuals/households.

When the transfer sensitivity of the Bonferroni coefficient is judged according to the principle of positional transfer sensitivity the results of Aaberge (2000) show that the Bonferroni coefficient (C_1) always treats a given transfer of money from a richer to a poorer person to be more equalizing the lower it occurs in the income distribution, provided that the difference in ranks between receivers and donors is the same.

For a discussion of the transfer sensitivity properties of the Gini coefficient (C_2) and the C_3-coefficient we refer to Aaberge (2000). However, for the sake of completeness we summarize the transfer sensitivity properties of the members of Gini's nuclear family of inequality measures in Proposition 1.

---

[16] For a complete characterization of logconcavity, see An (1998).
[17] Note that the income gap is equal to the average poverty gap when x coincides with the poverty line.

PROPOSITION 1. The three members of Gini's Nuclear Family, C_1, C_2 and C_3, have the following transfer sensitivity properties,

(i) The Bonferroni coefficient (C_1) satisfies the principle of diminishing transfers for all strictly log-concave distribution functions and the principle of positional transfer sensitivity for all distribution functions.

(ii) The Gini coefficient (C_2) satisfies the principle of diminishing transfers for all strictly concave distribution functions, but does not satisfy the principle of positional transfer sensitivity. In the case of a fixed difference in ranks the Gini coefficient attaches an equal weight to a given transfer irrespective of whether it takes place in the upper , the middle or the lower part of the income distribution.

(iii) The C_3-coefficient satisfies the principle of diminishing transfers for all distribution functions F for which F^2 is strictly concave, but does not satisfy the principle of positional transfer sensitivity. In the case of a fixed difference in ranks the C_3-coefficient assigns more weight to transfers at the upper than at the central and the lower parts of the income distribution.

As an empirical illustration of the methods proposed in this paper, Table 2 displays estimates of Gini's Nuclear Family with corresponding standard deviations[18] for the distribution of income after tax in Norway 1986 – 1998, where scale economies is accounted for by the use of the square root scale[19]. Exploring the trend in income inequality in this period is particularly interesting because a major tax reform was implemented in 1993, where taxation on capital income was substantially relaxed. Moreover, the Norwegian economy gradually recovered from a long recession at the end of 1992. Thus, we focus particular attention on the changes between 1986-1992 and 1993-1998. As is demonstrated by the estimates in Table 2, C_3 increased more than G and G more than C_1. Thus, according to the transfer sensitivity properties of C_1, G and C_3 indicated above, the rise in inequality is primarily due to increased inequality in the upper part of the income distribution. As suggested by Fjærli and Aaberge (2000) this result reflects the fact that changes in the tax reported dividends are the primary factor behind the changes in the standard reported inequality estimates and that most dividends are received by individuals located in the upper part of the income distribution.

---

[18] Methods for estimation and the asymptotic distribution theory for the empirical versions of the members of Gini's nuclear family are reported in Section 4.

[19] A computer program for estimating the scaled conditional mean curve and the measures of Gini's nuclear family as well as the related variances (standard deviations) is available on request. Note that the program allows for weighting of the observations when it is required due to the sampling design of the survey in question.

Table 2. Trend in income inequality in Norway, 1986-1998*

Year	C_1	C_2=G	C_3
1986	0.331 (0.002)	0.224 (0.002)	0.177 (0.002)
1987	0.330 (0.003)	0.224 (0.003)	0.177 (0.002)
1988	0.327 (0.003)	0.223 (0.002)	0.176 (0.002)
1989	0.340 (0.004)	0.233 (0.004)	0.186 (0.004)
1990	0.343 (0.003)	0.232 (0.002)	0.183 (0.002)
1991	0.340 (0.003)	0.232 (0.003)	0.185 (0.003)
1992	0.348 (0.003)	0.23 (0.003)	0.18 (0.002)
1993	0.352 (0.005)	0.240 (0.005)	0.191 (0.005)
1994	0.366 (0.003)	0.249 (0.002)	0.199 (0.002)
1995	0.358 (0.003)	0.247 (0.003)	0.198 (0.003)
1996	0.364 (0.004)	0.255 (0.004)	0.207 (0.004)
1997	0.371 (0.004)	0.260 (0.004)	0.212 (0.004)
1998	0.355 (0.003)	0.249 (0.003)	0.202 (0.003)
Average of (1986-92)	0.337 (0.001)	0.229 (0.001)	0.181 (0.001)
Average of (1993-98)	0.361 (0.002)	0.250 (0.001)	0.201 (0.001)
Percentage change, (1986-92) - (1993-1998)	7.14	9.07	10.96

Source: Fjærli and Aaberge (2000). *Standard deviation in parentheses

4. Estimation and asymptotic distribution results

Let $X_1, X_2, ..., X_n$ be independent random variables with common distribution function *F* and let F_n be the corresponding empirical distribution function. Moreover, let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denote the ordered $X_1, X_2, ..., X_n$. Since the parametric form of *F* is unknown, it is natural to use the empirical distribution function F_n to estimate *F* and to use

(17) $$M_n\left(\frac{i}{n}\right)=\frac{\frac{1}{i}\sum_{j=1}^{i}X_{(j)}}{\bar{X}},\quad i=1,2,...,n$$

to estimate $M(u)$ for $u=i/n$, where $\bar{X}$ is the sample mean.

By replacing *M* by M_n in the expression (8) for C_k, we get the following estimator[20] of the moments of the scaled conditional mean curve[21]

(18) $$\hat{C}_k=\hat{C}_k(F_n)=k\int_0^1 u^{k-1}\left(1-M_n(u)\right)du,\quad k=1,2,...$$

In order to derive the asymptotic distribution of the empirical rank-dependent measures of inequality it is convenient to firstly derive the asymptotic properties of the empirical scaled conditional mean curve *M*[22].

Approximations to the variance of M_n and the asymptotic properties of M_n can be obtained by considering the limiting distribution of the process $V_n(u)$ defined by

(19) $$V_n(u)=n^{\frac{1}{2}}\left[M_n(u)-M(u)\right].$$

Assume that the support of *F* is a non-empty finite interval $[a,b]$. (When *F* is an income distribution, *a* is commonly equal to zero.) Then $V_n(u)$ is a member of the space *D* of functions on [0,1] which are right continuous and have left hand limits. On this space we use the Skorokhod topology and the associated σ-field (e.g. Billingsley (1968), p. 111). We let $W_0(t)$ denote a Brownian Bridge on [0,1], that is, a Gaussian process with mean zero and covariance function $s(1-t),\ 0\le s\le t\le 1$. Moreover, let *Y(u)* be the Gaussian process defined by

(20) $$Y(u)=\frac{1}{u}\int_0^u\frac{W_0(t)}{f\left(F^{-1}(t)\right)}dt$$

[20] As demonstrated by Chotikapanich and Griffiths (2001) for the extended Gini coefficients, an alternative estimator performs better when the informational basis is restricted to group data with less than 20 groups.

[21] Since $M_n(\cdot)$ is a discrete function the integration symbol $\int$ represents numerical integration in this case.

[22] We refer to Goldie (1977) for an alternative proof of the asymptotic properties of the empirical Lorenz curve.

and let $\gamma^2(u)$ and $\kappa(u,v)$ be given by

(21) $$\gamma^2(u) = \frac{2}{u}\int_a^{F^{-1}(u)}\int_a^{y} F(x)\left(1-F(y)\right)dx\,dy, \quad 0 \le u \le 1$$

and

(22) $$\kappa(u,v) = \frac{1}{uv}\int_{F^{-1}(u)}^{F^{-1}(v)}\int_a^{F^{-1}(u)} F(x)\left(1-F(y)\right)dx\,dy, \quad 0 \le u \le v \le 1 .$$

The following result follows from Aaberge (1982, 2006),

THEOREM 2. Suppose that *F* has a continuous nonzero derivate *f* on $[a,b]$. Then $V_n(u)$ converges in distribution to the process

(23) $$V(u) = \frac{1}{\mu}\left[Y(u) - M(u)Y(1)\right],$$

with covariance function $\psi^2(u,v)$ given by

(24) $$\begin{aligned}\psi^2(u,v) = \frac{1}{\mu^2}\Big[&\frac{1}{v}\gamma^2(u) + \kappa(u,v) - M(u)\left(\gamma^2(v) + \kappa(v,1)\right)\\ &-M(v)\left(\gamma^2(u) + \kappa(u,1)\right) + M(u)M(v)\gamma^2(1)\Big], \quad 0 < u \le v \le 1.\end{aligned}$$

In order to construct confidence intervals for the scaled conditional mean curve at fixed points, we apply the results of *Theorem 2* which imply that the distribution of

$$n^{\frac{1}{2}}\frac{M_n(u) - M(u)}{\psi(u,u)}$$

tends to the $N(0,1)$ distribution for fixed *u*.

We shall now study the asymptotic distribution of the k-th order moment $\hat{C}_k$ (defined by (18)) of the empirical scaled conditional mean curve $\mathrm{M_n}(\cdot)$. As will be demonstrated below *Theorem 2* forms a helpful basis for deriving the asymptotic variance of $\hat{C}_k$.

Let $\theta_k{}^2$ be a parameter defined by

(25)
$$\theta_k{}^2 = \frac{1}{\mu^2}\left\{2k^2 \int_0^1 \int_0^v \left[\frac{1}{v}\gamma^2(u) + \kappa(u,v)\right](uv)^{k-1}\,du\,dv \right.$$
$$\left. -2[1-C_k]\left[k\int_0^1 \left(\gamma^2(u)+\kappa(u,1)\right)u^{k-1}du\right] + \gamma^2(1)[1-C_k]^2\right\}.$$

THEOREM 3. Suppose the conditions of *Theorem 2* are satisfied and $\theta_k{}^2 < \infty$. Then the distribution of

$$n^{\frac{1}{2}}\left(\hat{C}_k - C_k\right)$$

tends to the normal distribution with zero mean and variance $\theta_k{}^2$.

PROOF. From (8), (18) and (19) we see that

$$n^{\frac{1}{2}}\left(\hat{C}_k - C_k\right) = -k\int_0^1 u^{k-1}V_n(u)du\ .$$

By *Theorem 2* we have that $V_n(u)$ converges in distribution to the Gaussian process $V(u)$ defined by (23). By applying Billingsley (1968, Theorem 5.1) and Fubini's theorem we get that $n^{\frac{1}{2}}\left(\hat{C}_k - C_k\right)$ converges in distribution to

$$-k\int_0^1 u^{k-1}V(u)du = -k\int_0^1 u^{k-1}\left(\sum_{j=1}^{\infty} d_j(u)Z_j\right)du = -\sum_{j=1}^{\infty}\left[k\int_0^1 u^{k-1}d_j(u)du\right]Z_j$$

where $Z_1, Z_2, \ldots$ are independent $N\left(0,1\right)$ variables and $d_j(u)$ is defined by

(26)
$$d_j(u) = \frac{1}{\mu}\left[p_j(u) - p_j(1)M(u)\right]$$

and $p_j(u)$ is defined by

(27)
$$p_j\left(u\right) = \frac{2^{\frac{1}{2}}}{j\pi u}\int_0^u \frac{\sin\left(j\pi t\right)}{f\left(F^{-1}(t)\right)}dt\ ,$$

i.e., the asymptotic distribution of $n^{\frac{1}{2}}\left(\hat{C}_k - C_k\right)$ is normal with mean zero and variance

(28) $$\sum_{j=1}^{\infty}\left[k\int_0^1 u^{k-1}d_j(u)du\right]^2.$$

Then it remains to show that the asymptotic variance is equal to $\theta_k{}^2$.

Inserting (26) in (28), we get

$$\sum_{j=1}^{\infty}\left[k\int_0^1 u^{k-1}d_j(u)du\right]^2=\frac{1}{\mu^2}\sum_{j=1}^{\infty}\left[k\int_0^1 u^{k-1}\left(p_j(u)-p_j(1)M(u)\right)du\right]^2$$
$$=\frac{1}{\mu^2}\left\{\sum_{j=1}^{\infty}\left[k\int_0^1 u^{k-1}p_j(u)du\right]^2-2\left[k\int_0^1 u^{k-1}M(u)du\right]\left[\sum_{j=1}^{\infty}p_j(1)k\int_0^1 u^{k-1}p_j(u)du\right]\right.$$
$$\left.+\left[\sum_{j=1}^{\infty}p_j^2(1)\right]\left[k\int_0^1 u^{k-1}M(u)du\right]^2\right\}.$$

In the following derivation we apply Fubini's theorem and the identity

(29) $$2\sum_{j=1}^{\infty}\frac{\sin(j\pi s)\sin(j\pi t)}{(j\pi)^2}=s(1-t),\quad 0\le s\le t\le 1.$$

$$\sum_{j=1}^{\infty}\left[k\int_0^1 u^{k-1}p_j(u)du\right]^2=\sum_{j=1}^{\infty}k^2\int_0^1\int_0^1(uv)^{k-2}\,p_j(u)p_j(v)du\,dv$$
$$=k^2\int_0^1\int_0^1\left[\int_0^v\int_0^u\frac{2}{f\left(F^{-1}(t)\right)f\left(F^{-1}(s)\right)}\left(\sum_{j=1}^{\infty}\frac{\sin(j\pi t)\sin(j\pi s)}{(j\pi)^2}\right)dt\,ds\right](uv)^{k-2}\,du\,dv$$
$$=2\int_0^1\int_0^v k^2\left[2\int_0^u\int_0^s\frac{t(1-s)}{f\left(F^{-1}(t)\right)f\left(F^{-1}(s)\right)}dt\,ds+\int_u^v\int_0^u\frac{t(1-s)}{f\left(F^{-1}(t)\right)f\left(F^{-1}(s)\right)}dt\,ds\right](uv)^{k-2}\,du\,dv$$
$$=2\int_0^1\int_0^v(uv)^{k-2}k^2\left[2\int_a^{F^{-1}(u)}\int_a^y F(x)\left(1-F(y)\right)dx\,dy+\int_{F^{-1}(u)}^{F^{-1}(v)}\int_a^{F^{-1}(u)}F(x)\left(1-F(y)\right)dx\,dy\right]du\,dv$$
$$=2\int_0^1\int_0^v k^2\,(uv)^{k-1}\left[\frac{1}{v}\gamma^2(u)+\kappa(u,v)\right]du\,dv$$

where $\gamma^2(u)$ and $\kappa(u,v)$ are given by (21) and (22), respectively. Similarly, we find that

$$\sum_{j=1}^{\infty}p_j(1)\int_0^1 u^{k-2}p_j(u)\,du=\int_0^1 k\left[\gamma^2(u)+\kappa(u,1)\right]u^{k-1}du.$$

By noting that

$$\sum_{j=1}^{\infty} p_j(1) = \gamma^2(1)$$

and that

$$k\int_0^1 u^{k-1}M(u)du = 1 - C_k ,$$

the proof is completed.

Q.E.D.

For k=2, *Theorem 3* states that $\theta_2^{\,2} = \alpha^2$, where α^2 is defined by

$$(30) \quad \alpha^2 = \frac{4}{\mu^2}\left\{2\int_0^1\int_0^v u\left[\gamma^2(u) + v\kappa(u,v)\right]du\,dv - (1-G)\int_0^1 u\left[\gamma^2(u) + \kappa(u,1)\right]du + \frac{1}{4}(1-G)^2\gamma^2(1)\right\},$$

is the asymptotic variance of the empirical Gini coefficient $n^{\frac{1}{2}}\hat{G}$ where G=C_2 and $\hat{G} = \hat{C}_2$ [23].

For k=1, *Theorem 3* provides the asymptotic variance β^2 of the empirical Bonferroni coefficient $n^{\frac{1}{2}}\hat{C}_1$

$$(31) \quad \beta^2 = \frac{1}{\mu^2}\left\{2\int_0^1\int_0^v \left[\frac{1}{v}\gamma^2(u) + \kappa(u,v)\right]du\,dv - 2(1-C_1)\int_0^1\left[\gamma^2(u) + \kappa(u,1)\right]dudv + (1-C_1)^2\gamma^2(1)\right\}$$

The estimation of $\theta_k^{\,2}$ is straightforward. As in Sections 2 and 3 we assume that the parametric form of *F* is not known. Thus, replacing *F* by the empirical distribution function F_n in expressions (21) and (22) for $\gamma^2(u)$ and $\kappa(u,v)$ and next by replacing C_k , μ , $\gamma^2(u)$ and $\kappa(u,v)$ by their respective estimates in expression (25) for $\theta_k^{\,2}$, we obtain a consistent nonparametric estimator for $\theta_k^{\,2}$.

[23] An alternative version of (30) is given by Hoeffding (1948).

5. Conclusion

This paper proposes to use a specific transformation of the Lorenz curve, called the scaled conditional mean curve, rather than the Lorenz curve as basis for choosing a few summary measures of inequality for empirical applications. The scaled conditional mean curve turns out to possess several attractive properties as an alternative interpretation of the information content of the Lorenz curve and furthermore proves to provide essential information on polarization in the population. The discussion in Section 3 demonstrates that the inequality measures C_1, C_2 and C_3 define the first three moments of the scaled conditional mean curve. Thus, jointly they may give a good summarization of inequality in the scaled conditional mean curve and consequently act as primary quantities for measuring inequality in distributions of income. Moreover, since C_2 is the Gini coefficient and C_1 and C_3 prove to supplement the Gini coefficient with regard to focus on the lower and the upper part of the income distribution, it should be justified to call the group formed by these three inequality measures the Gini's Nuclear Family. The paper also provides asymptotic distribution results for the empirical scaled conditional mean curve and the related family of empirical measures of inequality, including Gini’s nuclear family.

References

Aaberge, R. (1982): Om måling av ulikskap, Rapporter 82/9, Statistisk sentralbyrå (In Norwegian).

Aaberge, R. (2000): Characterizations of Lorenz curves and income distributions, *Social Choice and Welfare*, **17**, 639-653.

Aaberge, R. (2001): Axiomatic characterization of the Gini coefficient and Lorenz curve orderings, *Journal of Economic Theory*, **101**, 115-132.

Aaberge, R. (2004): Ranking Intersecting Lorenz Curves, CEIS Tor Vergata Research Paper Series, Discussion Paper No. 45.

Aaberge, R. (2006): Asymptotic distribution theory of empirical rank-dependent measures of inequality. In V. Nair (Ed.): *Advances in Statistical Modeling and Inference - Essays in Honor of Kjell A. Doksum*, World Scientific, 2006.

Aaberge, R., U. Colombino and S. Strøm (2004): Do more equal slices shrink the cake?: An empirical evaluation of tax-transfer reform proposals in Italy, *Journal of Population Economics*, **17**, 767 - 785.

An, M. Y. (1998): Logconcavity versus logconvexity: A complete characterization, *Journal of Economic theory*, **80**, 350-369.

Atkinson, A. B. (1979): On the measurement of inequality, *Journal of Economic Theory*, **2**, 244-263.

Atkinson, A. B. and F. Bourguignon (1989): The design of direct taxation and family benefits. *Journal of Public Economics*, **41**, 3-29.

Ben Porath, E. and I. Gilboa (1994): Linear measures, the Gini index, and the income-equality trade-off, *Journal of Economic Theory*, **64**, 443-467.

Billingsley, P. (1968): *Convergence of Probability Measures*. John Wiley & Sons, Inc., New York.

Bonferroni, C. (1930): *Elementi di Statistica Generale*. Seeber, Firenze.

Bossert, W. (1990): An approximation of the single-series Ginis, *Journal of Economic Theory*, **50**, 82-92.

Caplin, A. and B. Nalebuff (1991): Aggregation and social choice: A mean voter theorem, *Econometrica*, **59**, 1-23.

Chakravarty, S. R. and P. Muliere (2003): Welfare indicators: a review and new perspectives, *Metron*, **61**, 1-41.

Chotikapanich, D. and W. Griffiths (2001): On Calculation of the extended Gini coefficient, *Review of Income and wealth*, **47**, 541-547.

D'Addario, R. (1936): Curve di concentrazione, elasticitá, flessibilitá, densitá media e densitá marginale dei redditi, Cressati, Bari.

Donaldson, D. and J. A. Weymark (1980): A single parameter generalization of the Gini indices of inequality, *Journal of Economic Theory*, **22**, 67-86.

Donaldson, D. and J. A. Weymark (1983): Ethically flexible indices for income distributions in the continuum, *Journal of Economic Theory*, **29**, 353-358.

Éltetö, Ö. and E. Frigyes (1968): New inequality measures as efficient tools for causal analysis and planning, *Econometrica*, **36**, 383-396.

Esteban, J. M. and D. Ray (1994): On the measurement of polarization, *Econometrica*, **62**, 819-851.

Fields, G. E. and J. C. H. Fei (1978): On inequality comparisons, *Econometrica*, **46**, 303-316.

Fjærli, E. and R. Aaberge (2000): Tax reforms, dividend policy and trends in income inequality: empirical evidence based on Norwegian data, Discussion Paper No. 284, Statistics Norway.

Giorgi, G. M. (1984): A methodological survey of recent studies for the measurement of inequality of economic welfare carried out by some Italian statisticians, *Economic Notes*, **13**, 145-157.

Giorgi, G. M. (1990) Bibliographic portrait of the Gini concentration ratio, *Metron*, **48**, 183-221.

Giorgi, G. M. (1998) Concentration index, Bonferroni; in Encyclopedia of Statistical Sciences, *Update*, **2**, 141-146.

Goldie, C. M. (1977).Convergence theorems for empirical Lorenz curves and their inverses, *Advances in Applied Probability*, ***9***, 765-791.

Hardy, G. H., J.E. Littlewood and G. Polya (1934): “Inequalities”, Cambridge University Press, Cambridge.

Heckman, J. J. and B. Honoré (1990): The empirical content of the Roy model, *Econometrica*, **58**, 1121-1150.

Hey, J. D. and P. J. Lambert (1980): Relative deprivation and the Gini coefficient: comment, *Quarterly Journal of Economics*, **94**, 567-573.

Hoeffding, W. (1948): A Class of Statistics with Asymptotically Normal Distribution. *The Annals of Mathematical Statistics*, **19**, 293-325.

Kolm, S. C. (1969): The optimal production of social justice, in “Public Economics”, (J. Margolis and H. Guitton, Eds.), Macmillan, New York/London.

Kolm, S. C. (1976): Unequal inequalities I, *Journal of Economic Theory*, **12**, 416-442.

Kolm, S. C. (1976): Unequal inequalities II, *Journal of Economic Theory*, **13**, 82-111.

Mehran, F. (1976): Linear measures of inequality, *Econometrica*, **44**, 805-809.

Nygård, F. and A. Sandström (1981): *Measuring Income Inequality*. Almqvist and Wiksell International, Stockholm.

Rothschild, M. and J. E. Stiglitz (1970): Increasing risk: A definition, *Journal of Economic Theory*, **2**, 225-243.

Sen, A. (1974): Informational bases of alternative welfare approaches, *Journal of Public Economics*, **3**, 387-403.

Weymark, J. (1981): Generalized Gini inequality indices, *Mathematical Social Sciences*, **1**, 409-430.

Wolfson, M. (1994): When inequalities diverge, *American Economic Review*, **84**, 353-358.

Yaari, M.E. (1987): The dual theory of choice under risk, *Econometrica*, **55**, 95-115.

Yaari, M.E. (1988): A controversial proposal concerning inequality measurement, *Journal of Economic Theory*, **44**, 381-397.

Yitzhaki, S. (1983): On an extension of the Gini inequality index, *International Economic Review*, **24**, 617-628.

Zoli, C. (1999): Intersecting generalized Lorenz curves and the Gini index, *Social Choice and Welfare*, **16**, 183-196.

Figure 1. Lorentz curves for distributions of average annual earnings in Norway

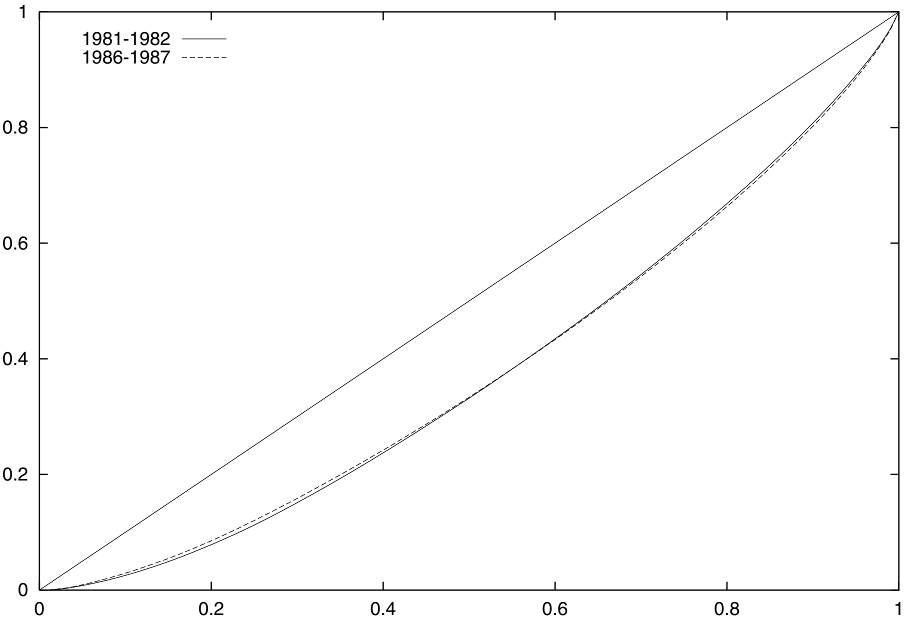

Figure 2. Scaled conditional mean curves for distributions of average annual earnings in Norway