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# Axiomatic Characterization of the Gini Coefficient and Lorenz Curve Orderings

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## Abstract

This paper is concerned with distributions of income and the ordering of related Lorenz curves. By introducing appropriate preference relations on the set of Lorenz curves, two alternative axiomatic characterizations of Lorenz curve orderings are proposed. Moreover, the Gini coefficient is recognized to be rationalizable under both axiom sets; as a result, a complete axiomatic characterization of the Gini coefficient is obtained. Furthermore, axiomatic characterizations of the extended Gini family and an alternative "generalized" Gini family of inequality measures are proposed.

**Keywords:** Lorenz curve orderings, axiomatic characterization, measures of inequality, the Gini coefficient.

**JEL classification:** D31, D63.

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## 1. Introduction

Analyses of income distribution focus both on the level of income and relative income differences, i.e. on the size and the division of the cake. In applied work the standard approach is to separate these two dimensions and use the Lorenz curve as a basis for analysing the relative income differences. By displaying the deviation of each individual income share from the income share that corresponds to perfect equality, the Lorenz curve captures the essential descriptive features of the concept of inequality. The normative aspects of Lorenz curve orderings have been discussed by Kolm (1969, 1976a, 1976b), Sen (1973), and Atkinson (1970) who demonstrated that Lorenz curve orderings of distributions with equal means may correspond to social welfare orderings. The assumption of equal means, however, limits the applicability of their results. Real world interventions that alter the income distribution are usually not mean preserving changes; taxes and transfer programs, for example, are interventions that decrease and increase the mean level of income. If, in such situations, concern about inequality is over relative rather than absolute income differences[1], the condition of scale invariance has to be introduced. The condition of scale invariance implies that inequality is compatible with the representation given by the Lorenz curve. Thus, adopting the Lorenz curve as a basis for judging between income distributions means that we are only concerned about the distributional aspects independent of the level of mean income. This is in line with common practice in applied economics where the Lorenz curve and related summary measures of inequality are used to compare inequality in distributions with different mean incomes[2]. This practice demonstrates that the ordering of Lorenz curves in cases of variable mean income is of interest in its own right, irrespective of how we judge a possible conflict between the level of mean income and the degree of (in)equality and its implications for social welfare.

In theories of social welfare it has long been considered very important to decompose social welfare with respect to mean income and inequality. The standard approach is to derive this decomposition and the related measures of inequality from specified social welfare functions, see e.g. Kolm (1969), Atkinson (1970), Sen (1973), and Blackorby and Donaldson (1978) who also proposed a method for deriving social welfare functions from given measures of inequality. For a critical discussion of the Kolm-Atkinson approach see Sen (1978), and Ebert (1987) who also introduced an alternative approach by explicitly taking into account value judgments of the trade-off between mean and (in)equality in deriving social welfare functions. As a first step, Ebert (1987) introduced a mean-independent ordering of income distributions in terms of inequality. Then, to deal with the mean-equality trade-off an ordering was defined on pairs of mean income and degrees of inequality.

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[1] The importance of focusing on relative incomes has been acknowledged since ancient times and was e.g. discussed by Plato who proposed that the ratio of the top income to the bottom should be less than four to one (see Cowell, 1977). See also Sen's discussion of relative deprivation in the space of income and Smith's (1776) discussion of necessities.

Moreover, the second ordering was required to be consistent with the first ordering in the sense that the distribution with lowest inequality is preferred in comparisons of distributions with equal means. Ebert (1987) demonstrated that by combining these two orderings a social welfare ordering is obtained and furthermore that the related social welfare functions allow a mean-inequality split-up[3].

Now, referring to the standard practice of separately comparing the means and Lorenz curves of income distributions it appears attractive to represent the distribution by the mean income and the Lorenz curve. Thus, following Ebert's two-step approach, orderings defined on Lorenz curves can be used as a basis for deriving social welfare orderings and related welfare functions. The starting point is to introduce a preference ordering on the set of Lorenz curves as a basis for assessing the degree of inequality. Next, combining this ordering with a second ordering defined on pairs of mean income and degrees of inequality, a social welfare ordering is obtained.

The purpose of this paper is to provide an axiomatic basis for Lorenz curve orderings. Judgments concerning trade-off between mean and (in)equality are, however, beyond the scope of the paper. Section 2 presents two alternative sets of assumptions concerning a person's preferences over Lorenz curves and gives convenient representations of the corresponding preference relations. Furthermore, complete axiomatizations of the Gini coefficient, the extended Gini family and a new "generalized" Gini family of inequality measures are proposed. Section 3 introduces an alternative characterization of first-degree Lorenz dominance as a criterion for inequality aversion to those provided by Atkinson (1970) and Yaari (1988).

## 2. Representation results

In this section we shall demonstrate that the problem of ranking Lorenz curves can, formally, be viewed as analogous to the problem of choice under uncertainty. In theories of choice under uncertainty, preference orderings over probability distributions are introduced as basis for deriving utility indexes. In the present context the corresponding point of departure is to assume appropriate preference relations on the set of Lorenz curves.

The Lorenz curve L for a cumulative income distribution F with mean $\mu$ is defined by

(1) $$L(u) = \frac{1}{\mu} \int_{F(x) \leq u} x \, dF(x), \quad 0 \leq u \leq 1.$$

L is an increasing convex function with range [0,1]. Thus, L can be considered analogous to a convex distribution function on [0,1].

[2] See e.g. Atkinson (1970), Coder et al. (1989) and Atkinson et al. (1995) who make intercountry comparisons of Lorenz curves allowing for differences between countries in level of income.
[3] See also Ben Porath and Gilboa (1994) who proposed an alternative approach for dealing with the mean-equality approach.

Now, let **L** denote the family of Lorenz curves. Note that a convex combination of Lorenz curves is a Lorenz curve and hence a member of **L**. A person's ranking of elements from **L** may be represented by a preference relation $\succeq$, which will be assumed to satisfy the following basic axioms,

**Axiom 1** (Order). $\succeq$ *is a transitive and complete ordering on* ***L****.*

**Axiom 2** (Dominance). *Let* $L_1, L_2 \in \boldsymbol{L}$. *If* $L_1(u) \geq L_2(u)$ *for all* $u \in [0,1]$ *then* $L_1 \succeq L_2$.

**Axiom 3** (Continuity). *For each* $L \in \boldsymbol{L}$, *the sets* $\left\{L^* \in \boldsymbol{L} : L \succeq L^*\right\}$ *and* $\left\{L^* \in \boldsymbol{L} : L^* \succeq L\right\}$ *are closed (w.r.t.* $L_1$*-norm).*

Then it follows by Debreu (1964) that $\succeq$ can be represented by a continuous and increasing preference functional V from **L** to R. Hence, V defines an ordering on the set of all Lorenz curves. In order to give the order relation $\succeq$ an empirical content it is necessary to impose further restrictions on V. We can obtain convenient and testable representations of $\succeq$ by introducing appropriate independence conditions. Specifically, we shall consider the following two axioms, where $L^{-1}$ is the left inverse of L, $L^{-1}(0)=0$ and $L^{-1}(1)=1$.

**Axiom 4** (Independence). *Let* $L_1$*,* $L_2$ *and* $L_3$ *be members of* ***L*** *and let* $\alpha \in [0,1]$. *Then* $L_1 \succeq L_2$ *implies* $\alpha L_1 + (1-\alpha) L_3 \succeq \alpha L_2 + (1-\alpha) L_3$.

**Axiom 5** (Dual independence). *Let* $L_1$*,* $L_2$ *and* $L_3$ *be members of* ***L*** *and let* $\alpha \in [0,1]$. *Then* $L_1 \succeq L_2$ *implies* $\left(\alpha L_1^{-1} + (1-\alpha) L_3^{-1}\right)^{-1} \succeq \left(\alpha L_2^{-1} + (1-\alpha) L_3^{-1}\right)^{-1}$.

Axioms 4 and 5 correspond to the independence axioms of the expected utility theory and Yaari's dual theory of choice under uncertainty, respectively (see Yaari (1987)) and require that the ordering is invariant with respect to certain changes in the Lorenz curves being compared. If $L_1$ is weakly preferred to $L_2$, then Axiom 4 states that any mixture on $L_1$ is weakly preferred to the corresponding mixture on $L_2$. The intuition is that identical mixing interventions on the Lorenz curves being compared do not affect the ranking of Lorenz curves; the ranking depends solely on how the differences between the mixed Lorenz curves are judged.

In order to clarify the interpretation of Axiom 4, consider an example with a tax/transfer intervention that alters the shape of the income distributions and leaves the mean incomes unchanged: Let $F_1$ and $F_2$ be income distributions with means $\mu_1$ and $\mu_2$ and Lorenz curves $L_1$ and $L_2$. Now suppose that these distributions are affected by the following tax/transfer reform. First, a proportional tax with tax rate 1-$\alpha$ is introduced. Second, the collected taxes are in both cases redistributed according to appropriate scale transformations of some distribution function $F_3$ with mean $\mu_3$. This means that the two sets of collected taxes are redistributed according to the same Lorenz curve. It is understood that this redistribution is carried out so as to give equal-sized transfers or transfers that are less progressive than a set of equal-sized transfers. Specifically, this means that $(1-\alpha)(\mu_i/\mu_3)F_3^{-1}(t)$ is the transfer received by the t-quantile unit of the income distribution $F_i$. At the extreme, when $F_3$ is a degenerate distribution function, the transfers are equal to the average tax (1-$\alpha$)$\mu_i$. At the other extreme $F_3$ will give all the collected tax to the best well-off unit. After this tax/transfer intervention, the inverses of the two income distributions are given by

(2) $$\alpha F_i^{-1}(t)+(1-\alpha)\mu_i\frac{F_3^{-1}(t)}{\mu_3},\quad i=1,2,$$

where $F_i^{-1}(t)$ is the left inverse of $F_i$. Now, it follows readily from (2) that this intervention leaves the mean incomes, $\mu_1$ and $\mu_2$, unchanged. Moreover, (2) implies that the Lorenz curve for $F_i$ after the intervention have changed from $L_i$ to

(3) $$\frac{1}{\mu_i}\int_0^u\left[\alpha F_i^{-1}(t)+(1-\alpha)\mu_i\frac{F_3^{-1}(t)}{\mu_3}\right]dt=\alpha L_i(u)+(1-\alpha)L_3(u),\quad i=1,2.$$

Hence, if $L_1$ is weakly preferred to $L_2$, then Axiom 4 states that the changes in $L_1$ and $L_2$ that follows from the above intervention will not affect the ranking of Lorenz curves.

Axiom 5 postulates a similar invariance property on the inverse Lorenz curves to that postulated by Axiom 4 on the Lorenz curves. The essential difference between the two axioms is that Axiom 5 deals with the relationship between given income shares and weighted averages of corresponding population shares, while Axiom 4 deals with the relationship between given population shares and weighted averages of corresponding income shares. Thus, Axiom 5 requires the ordering relation $\succeq$ to be invariant with respect to aggregation of sub-populations across cumulative income shares. That is, if for a specific population the Lorenz curve $L_1$ is weakly preferred to the Lorenz curve $L_2$, then mixing this population with any other population with respect to the distributions of their income shares does not affect the ranking of Lorenz curves. As an illustration, consider a population divided into a group of poor and a group of rich where each unit's income is equal to the corresponding group mean. In judging between two-points distributions a person who approves Axiom 4 and disapproves Axiom 5 will be more concerned about the number of poor rather than about how poor they are. By contrast, a person who approves Axiom 5 and disapproves Axiom 4 will emphasize the size of the poor's income share rather than how many they are.

By restricting to distributions with equal means we see that Axiom 4 can be interpreted as a weaker version of Yaari's dual independence axiom. This means that a person who approves Yaari's dual independence axiom will always approve Axiom 4. Thus, the result in Theorem 1 can be considered as an alternative (and slightly different) version of the representation result of Yaari (1987, 1988).

THEOREM 1. *A preference relation* $\succeq$ *on* ***L*** *satisfies Axioms 1-4 if and only if there exists a continuous and non-increasing real function p(·) defined on the unit interval, such that for all* $L_1, L_2 \in \mathbf{L}$,

$$L_1 \succeq L_2 \Leftrightarrow \int_0^1 p(u)\, d\, L_1(u) \geq \int_0^1 p(u)\, d\, L_2(u). \tag{4}$$

*Moreover, p is unique up to a positive affine transformation.*

Proof. Assume that there exists a continuous and non-increasing real function p(·) such that (4) is true for all $L_1, L_2 \in \mathbf{L}$. Then by noting that

$$\int p(t)\, d\big(L_1(t) - L_2(t)\big) = -\int \big(L_1(t) - L_2(t)\big)\, dp(t),$$

it follows by straightforward verification that $\succeq$ satisfies Axioms 1-4.

To prove sufficiency, note that **L** is a subfamily of distribution functions. Furthermore, it follows from Axioms 1-4 that the conditions of Theorem 3 of Fishburn (1982) are satisfied and thus that there exists a continuous function p(·) satisfying (4) where p(·) is unique up to a positive affine transformation. It follows from the monotonicity property of Axiom 2 that p(·) is nonincreasing.

Q.E.D.

Now, let $V_P$ be a functional, $V_P : \mathbf{L} \to [0,1]$, defined by

$$V_P(L) = \int_0^1 P'(u)\, d\, L(u), \tag{5}$$

where P' is the derivative of P, which is a continuously differentiable and concave distribution function defined on the unit interval. Theorem 1 demonstrates that a person who supports Axioms 1-4 will rank Lorenz curves according to the criterion $V_P$. For convenience, and with no loss of generality, we assume $P'(1) = 0$. This is a normalization condition which ensures that $V_P$ has the unit interval as its range, taking the maximum value 1 if incomes are equally distributed and the minimum value 0 if one unit holds all income. Thus, $J_P$ defined by

(6) $$J_P(L) = 1 - \int_0^1 P'(u)\, dL(u),$$

measures the extent of inequality in an income distribution with Lorenz curve L, when P is the chosen preference or weight function[4]. By choosing $P(t) = 2t - t^2$ it follows directly from (6) and Theorem 1 that the Gini coefficient is rationalizable under Axioms 1-4. Note that $\mu V_P$ corresponds to the utility representation of the "dual theory" of choice under risk proposed by Yaari (1987, 1988) who also demonstrated that the absolute Gini difference was rationalizable under the "dual theory". Moreover, by choosing appropriate preference functions we can derive attractive alternatives to the Gini coefficient which are consistent with Theorem 1. For example, by choosing the following family of P-functions,

(7) $$P_k(t) = 1 - (1-t)^{k+1}, \quad k \geq 0,$$

we obtain the following family of measures of inequality

(8) $$G_k(L) = 1 - k(k+1)\int_0^1 (1-u)^{k-1} L(u)\,du, \quad k \geq 0.$$

The family $\{G_k\}$ is the extended Gini family of inequality measures, introduced by Donaldson and Weymark (1980) and by Kakwani (1980) as an extension of a poverty measure proposed by Sen (1976). For a discussion of the extended Gini family, we refer to Donaldson and Weymark (1980, 1983) and Yitzhaki (1983). By exploiting the fact that the Lorenz curve can be considered analogous to a cumulative distribution function Aaberge (2000) demonstrated that the subfamily of the extended Gini family formed by the integer values of k in (8) uniquely determines the Lorenz curve.[5] This means that no information is lost when we restrict attention to the integer subscript subfamily of the extended Gini family of inequality measures. Further discussion of the relationship between the integer subscript subfamily of the extended Gini family and the Lorenz curve is left for the next section.

As indicated above Axiom 4 is closely related to Yaari's dual independence axiom and can thus be considered to be an alternative of the independence axiom that forms the basis of the expected

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[4] Mehran (1976) introduced an alternative version of (6) based on descriptive arguments. For alternative normative motivations of the $J_P$-family and various subfamilies of the $J_P$-family we refer to Donaldson and Weymark (1980, 1983), Weymark (1981) and Ben Porath and Gilboa (1994).

[5] Actually, Aaberge (2000) demonstrated that the integer subscript subfamily of the extended Gini family is uniquely determined by a family of inequality measures which is formed by the moments of the Lorenz curve.

utility theory for choice under uncertainty. Another alternative is provided by Axiom 5 which postulates independence in terms of the inverse Lorenz curve. While the Lorenz curve only can be considered analogous to a distribution function its inverse actually proves to be a distribution function. This fact follows from the following expression

$$(9) \qquad L^{-1}(u) = F\left(K^{-1}(u)\right), \; 0 \le u \le 1,$$

where K defined by

$$(10) \qquad K(y) = \frac{1}{\mu}\int_0^y x \, dF(x)$$

is the first-moment distribution. Thus, $L^{-1}$ is the cumulative distribution of K(Y) where Y is a random variable with cumulative distribution function F. By aggregating the ordered incomes, starting with the lowest, up to $K^{-1}(u)$ 100u per cent of the total income is included. Hence, $L^{-1}(u)$ can be interpreted as the probability that a randomly drawn income from F is equal to or lower than $K^{-1}(u)$. Expression (10) of $L^{-1}$ demonstrates that Axiom 5 postulates a similar invariance on the transformed income distributions as the conventional independence axiom of the expected utility theory postulates on the income distributions. The essential difference between these axioms emerges in how they treat the introduction of a tax reform that exclusively concerns a certain region of a nation. Whilst the conventional independence axiom requires the judgment of this reform to be independent of the tax structure and distribution of incomes in the remaining part of the country, Axiom 5 assumes that the judgment of the reform depend on the income structure outside the region as well as on the distributional consequences of the reform.

Now, by replacing Axiom 4 with Axiom 5 in Theorem 1, we obtain the following alternative representation result.

THEOREM 2. *A preference relation $\succeq$ on **L** satisfies Axioms 1-3 and Axiom 5 if and only if there exists a continuous and non-decreasing real function q(·) defined on the unit interval, such that for all* $L_1, L_2 \in \boldsymbol{L}$,

$$(11) \qquad L_1 \succeq L_2 \Leftrightarrow \int_0^1 q\left(L_1(u)\right) du \ge \int_0^1 q\left(L_2(u)\right) du .$$

*Moreover, q is unique up to a positive affine transformation.*

Proof. It follows from (1) that there is a one-to-one correspondence between the Lorenz curve and its inverse. Hence, the ordering relation $\succeq$ defined on the set of inverse Lorenz curves is equivalent to the ordering relation defined on **L**. Note that $L_1^{-1}(u) \le L_2^{-1}(u)$ for all $u \in [0,1]$ if and only if $L_1(u) \ge L_2(u)$ for all $u \in [0,1]$. Then, by replacing Axiom 4 with Axiom 5, Theorem 2 follows directly from Theorem 1 where the ordering representation is given by

$$\int_0^1 q(t)\, d\, L^{-1}(t) = \int_0^1 q\left(L(u)\right) du .$$

Q.E.D.

Now, let $V_Q^*$ be a functional, $V_Q^* : \mathbf{L} \to [0,1]$, defined by

(12) $$V_Q^*(L) = \int_0^1 Q'\left(L(u)\right) du = \int_0^1 Q'(t)\, d\, L^{-1}(t),$$

where Q' is the derivative of Q, a continuous and convex distribution function defined on the unit interval. It follows from Theorem 2 that $V_Q^*$ represents preferences that satisfy Axioms 1-3 and 5. The implication is that a person whose preferences satisfy Axioms 1-3 and 5 will choose among Lorenz curves so as to maximize $V_Q^*$. Further restrictions on the preferences can be introduced through the preference function Q. For normalization purposes we impose the condition $Q'(0) = 0$. This condition implies that $V_Q^*$ has the unit interval as its range, taking the maximum value 1 if incomes are equally distributed and the minimum value 0 if one unit holds all income. Thus, $J_Q^*$ defined by

(13) $$J_Q^*(L) = 1 - \int_0^1 Q'\left(L(u)\right) du$$

measures the extent of inequality in an income distribution with Lorenz curve L when social preferences are consistent with Axioms 1-3 and 5. By choosing $Q(t) = t^2$ in (11) it follows that $J_Q^*$ coincides with the Gini coefficient. Surprisingly, there seem to be no proposals of alternatives to the Gini coefficient that are consistent with Theorem 2. However, by specifying appropriate preference functions in (13) we can derive measures of inequality which are consistent with Theorem 2. For example, by introducing the following family of preference functions

(14) $$Q_k(t) = t^{k+1}, \quad k \ge 0,$$

we obtain the following related family of inequality measures

(15) $$G_k^*(L) = 1 - \left(k+1\right) \int_0^1 L^k(u)\, du, \quad k \ge 0,$$

where $G_1^*$ is the Gini coefficient.

As mentioned above $L^{-1}$ can be considered as a distribution function. Thus, the inverse Lorenz curve is uniquely determined by its moments, which means that the Lorenz curve can be recovered from the knowledge of $\{G_k^*(L): k = 1, 2, ...\}$ which we will denote the inverse Lorenz family of inequality measures. The inequality aversion properties of the $J_Q^*$-measures will be examined in the next section.

Note, however, that $J_Q^*$-measures can be viewed as a sum of weighted population shares, where the weights depend on the functional form of the Lorenz curve in question and thereby on the magnitude of the income shares. By contrast, $J_P$-measures can be viewed as a sum of weighted income shares, where the weights depend on population shares rather than income shares. Hence, these weights do not depend on the magnitudes of the income shares, but merely on the rankings of income shares. Now, let us return to the above discussion of two-points distributions in the context of Axioms 4 and 5. Note that the effect on $J_P$-measures of increasing the income share of the poor depends solely on the relative number of poor irrespective of their share of income, while a similar effect on $J_Q^*$-measures depends both on the poor's share of the population and their incomes. In contrast, the effect on $J_Q^*$-measures of an increase in the relative number of poor depends merely on the poor's share of the incomes, while the effect on $J_P$-measures depends both on the poor's share of population and income.

Postulating further conditions on the ordering relations, in addition to Axioms 1-4 and Axioms 1-3 and 5, respectively, will allow us to demonstrate that it is possible to obtain axiomatizations of the extended Gini family defined by (8) and the alternative "generalized" Gini family defined by (15). To this end it will be convenient to introduce the Lorenz curves $L_{a,b}$, defined by

$$(16) \qquad L_{a,b}(u) = \begin{cases} 0, & u < a \\ b\dfrac{u-a}{1-a}, & a \le u < 1, \; a \in [0,1) \text{ and } b \in [0,1] \\ 1, & u = 1 \end{cases}$$

and the related family $\tilde{\mathbf{L}}$ of these curves is defined formally by $\tilde{\mathbf{L}} = \{L : L = L_{a,b}, a \in [0,1) \text{ and } b \in [0,1]\}$. Note that $\tilde{\mathbf{L}}$ for $a, b \in \langle 0,1 \rangle$ represents distributions for which the population is divided into three groups, where one group (100a per cent of the population) has zero income, the second group contains just one unit holding 100 (1-b) per cent of the total income, while the units of the third group receive equal shares of the remaining 100b per cent of the income. Now, assuming that preferences consistent with Axioms 1-4 also satisfy Axiom 5 for all $L \in \tilde{\mathbf{L}}$ and alternatively that preferences consistent with Axioms 1-3 and 5 satisfy Axiom 4 for all $L \in \tilde{\mathbf{L}}$, the following representation results are obtained.

THEOREM 3. *A preference relation* $\succeq$ *on* $\boldsymbol{L}$ *satisfies Axioms 1-4 for all* $L \in \boldsymbol{L}$ *and Axiom 5 for all* $L \in \tilde{\boldsymbol{L}}$ *if and only if there exists a positive real number k such that for all* $L_1, L_2 \in \boldsymbol{L}$,

$$L_1 \succeq L_2 \Leftrightarrow \int_0^1 (1-u)^k \, d\, L_1(u) \geq \int_0^1 (1-u)^k \, d\, L_2(u).$$

*Moreover, the representation is unique up to a positive affine transformation.*

THEOREM 4. *A preference relation* $\succeq$ *on* $\boldsymbol{L}$ *satisfies Axioms 1-3 and 5 for all* $L \in \boldsymbol{L}$ *and Axiom 4 for all* $L \in \tilde{\boldsymbol{L}}$ *if and only if there exists a positive real number k such that for all* $L_1, L_2 \in \boldsymbol{L}$,

$$L_1 \succeq L_2 \Leftrightarrow \int_0^1 \left(L_1(u)\right)^k du \geq \int_0^1 \left(L_2(u)\right)^k du.$$

*Moreover, the representation is unique up to a positive affine transformation.*

Proof of Theorems 3 and 4. The necessary parts of Theorems 3 and 4 follow by straightforward verification. To prove sufficiency, note that Theorems 1 and 2 imply that there exists a continuous and non-increasing real function p(·), a continuous and non-decreasing real function q(·) and a monotone real function Ψ(·), such that

$$\text{(17)} \qquad \int_0^1 p(u)\, d\, L(u) = \Psi\left(\int_0^1 q\left(L(u)\right) du\right) \quad \text{for all} \quad L \in \tilde{\mathbf{L}}.$$

Hence, inserting for $L \in \tilde{\mathbf{L}}$ (17) equals

$$\text{(18)} \qquad \Psi\left((1-a)\frac{Q(b)}{b}\right) = b\frac{1-P(a)}{1-a} \quad \text{for all} \quad a \in [0,1) \text{ and } b \in [0,1].$$

where $Q(u) = \int_0^u q(t)\, dt$ and $P(u) = \int_0^u p(t)\, dt$. Without loss of generality we introduce the following suitable normalization, $P(0) = Q(0) = \Psi(0) = 0, P(1) = Q(1) = \Psi(1) = 1$ and $p(1) = q(0) = 0$. Then, (18) yields

$$\text{(19)} \qquad \Psi(1-a) = \frac{1-P(a)}{1-a} \quad \text{for all} \quad a \in [0,1),$$

and

(20) $$\Psi\left(\frac{Q(b)}{b}\right) = b \quad \text{for all} \quad b \in [0,1].$$

Now, inserting (19) and (20) in (18) we obtain the following equation

(21) $$\Psi\left((1-a)\frac{Q(b)}{b}\right) = \Psi(1-a)\,\Psi\left(\frac{Q(b)}{b}\right) \quad \text{for all} \quad a, b \in [0,1].$$

Note that (21) is equivalent to the functional equation

(22) $$\Psi(xy) = \Psi(x)\,\Psi(y) \quad \text{for all} \quad x, y \in [0,1]$$

which has the following solution (see e.g. Aczel, 1966), $\Psi \equiv 0$ or 1, or there exists $r > 0$ such that

(23) $$\Psi(x) = x^r \quad \text{for all} \quad x \in [0,1].$$

Inserting (23) into (19) and (20), respectively, the results of Theorems 3 and 4 are obtained.

Q.E.D.

In addition to providing a rationale for the extended Gini family **G** and the alternative "generalized" Gini family $\mathbf{G}^*$ of measures of inequality, Theorems 3 and 4 demonstrate that **G** and $\mathbf{G}^*$ are ordinally equivalent on $\tilde{\mathbf{L}}$.

Although several authors have discussed rationales for the absolute Gini difference (see Sen (1974), Hey and Lambert (1980), Weymark (1981) Donaldson and Weymark (1983) and Yaari (1987, 1988)), for the Gini coefficient (Thon (1982)) and for the absolute Gini differences as well as for the Gini coefficient (Ben Porath and Gilboa (1992)), no one has established a rationale for the Gini coefficient as a preference ordering on Lorenz curves. Moreover, a complete axiomatic characterization of the absolute Gini difference and/or the Gini coefficient has neither been provided. However, as was demonstrated above, the Gini coefficient is rationalizable under Axioms 1-4 as well as under Axioms 1-3 and 5. Thus, we may conjecture that the Gini coefficient represents the preference relation which satisfy Axioms 1-5.

THEOREM 5. *A preference relation $\succeq$ on **L** satisfies Axioms 1-5 if and only if $\succeq$ can be represented by the Gini coefficient.*

Proof. The necessary part of the theorem follows by straightforward verification. To prove the sufficiency part we observe from the proof of Theorem 3 and 4 that $\succeq$ satisfies Axioms 1-5 for all $L \in \tilde{\mathbf{L}}$ if and only if

(24) $$(r+1)\int_0^1 (1-u)^r \, dL(u) = \left[\left(1+\frac{1}{r}\right)\int_0^1 L^{1/r}(u)\,du\right]^r, \; r>0, \text{ for all } L \in \tilde{\mathbf{L}},$$

where equation (24) is established by inserting (16) in (17) and $\Psi$ is given by (23). Now, as (24) has to hold for all $L \in \mathbf{L}$ we obtain the following equation by inserting $L(u) = u^2$ in (24),

(25) $$\frac{2}{r+2} = \left(\frac{r+1}{r+2}\right)^r, \; r > 0.$$

By applying Lemma 1 in Appendix, it follows that equation (25) has a unique solution for $r = 1$.

Q.E.D.

Theorem 5 provides a complete axiomatization for the Gini coefficient. Thus, application of the Gini coefficient means that both independence axioms are supported jointly. Hence, the preferences of a person whose ethical norms coincide with the Gini coefficient are invariant with respect to certain types of tax/transfer interventions and with respect to aggregation of sub-populations across income shares.

REMARK. By restricting the comparison of Lorenz curves to distributions with equal means Axioms 1-5 can be considered to be defined on the set of generalized Lorenz curves rather on the set of Lorenz curves[6]. Thus, in this case the above representation results are valid for generalized Lorenz curves as well as for Lorenz curves.

When the preference relations in Theorems 1 and 2 are defined on the set of distribution functions rather on the set of Lorenz curves the representation results in Theorems 1 and 2 coincide with the conventional expected utility theory and Yaari's rank-dependent utility theory for choice under uncertainty. Thus, it follows from Theorem 5 that risk neutral behavior is completely characterized by Axioms 1-5 provided that these axioms are defined on the set of distribution functions **F**.

COROLLARY 1. *A preference relation* $\succeq$ *on* ***F*** *satisfies Axioms 1-5 if and only if* $\succeq$ *can be represented by the mean.*

[6]The generalized Lorenz curve, defined as a mean scaled-up version of the Lorenz curve, was introduced by Shorrocks (1983).

## 3. Inequality aversion

In expected utility theory it is standard to impose restrictions on the utility function applying various types of stochastic dominance rules. For example, "risk aversion" is equivalent to second-degree stochastic dominance and imposes strict concavity on the utility function. Analogous to the standard theory of choice under uncertainty Atkinson (1970) defined inequality aversion as being equivalent to risk aversion. This was motivated by the fact that the Pigou-Dalton transfer principle is identical to the principle of mean preserving spread introduced by Rothschild and Stiglitz (1970) which is equivalent to the condition of dominating non-intersecting Lorenz curves when we restrict attention to distributions with equal means. These principles may even be used as basis for discussing and interpreting cross-country comparisons of Lorenz curves. To perform inequality comparisons with Lorenz curves we can deal with distributions of relative incomes or alternatively simply abandon the assumption of equal means. The latter approach normally forms the basis of empirical studies and is also employed in this paper.

An interesting question is what kind of restrictions inequality aversion (dominating non-intersecting Lorenz curves) places on the preference functions P and Q? Yaari (1988) demonstrated that $J_P$ defined by (6) supports the criterion of dominating non-intersecting Lorenz curves if and only if P is strictly concave. An alternative characterization of the criterion of non-intersecting Lorenz curves is provided by Theorem 6.

Let $\mathbf{Q}_1$ be a class of preference functions related to $J_Q^*$ and defined by

$$\mathbf{Q}_1 = \left\{ Q : Q' \text{ and } Q'' \text{ are continuous on } [0,1],\ Q'(t) > 0 \text{ and } Q''(t) > 0 \text{ for } t \in \langle 0,1 \rangle, \text{ and } Q'(0) = 0 \right\}.$$

THEOREM 6. *Let $L_1$ and $L_2$ be members of **L**. Then the following statements are equivalent,*

*(i)* $L_1(u) \geq L_2(u)$ *for all* $u \in [0,1]$

*and the inequality holds strictly for at least one* $u \in \langle 0,1 \rangle$

*(ii)* $J_Q^*(L_1) < J_Q^*(L_2)$ *for all* $Q \in \mathbf{Q}_1$.

(Proof in Appendix.)

Theorem 6 demonstrates that first-degree Lorenz dominance and thereby inequality aversion is characterized by the strict convexity of Q-functions. Alternatively, it can be characterized by the strict concavity of P-functions. Based on these results one might expect that a "more concave" P-function or a "more convex" Q-function would exhibit more inequality aversion. As was recognized by Yaari (1988) and easily can be verified from (6) $J_{P_1}$ displays more inequality aversion than $J_{P_2}$ if $P_1$ lies above $P_2$ and $P_1$ and $P_2$ are concave. Similarly, it follows from (13) that $J^*_{Q_1}$ exhibits more inequality aversion than $J^*_{Q_2}$ if $Q_1$ is lying below $Q_2$ and $Q_1$ and $Q_2$ are convex.

Let $\mathbf{P}_1$ be a class of preference functions related to $J_P$ and defined by

$$\mathbf{P}_1 = \left\{ P : P' \text{ and } P'' \text{ are continuous on } [0,1],\ P'(t) > 0 \text{ and } P''(t) < 0 \text{ for } t \in \langle 0,1 \rangle, \text{ and } P'(1) = 0 \right\}.$$

An interesting question is whether the characterization of Lorenz dominance can be achieved for smaller classes of preference functions than $\mathbf{P}_1$ and $\mathbf{Q}_1$. As referred to above Aaberge (2000) demonstrated that the subfamily $\{G_k : k = 1, 2, ...\}$ of the extended Gini family uniquely determines the Lorenz curve. Thus, to characterize Lorenz dominance we may restrict attention to the family of P-functions defined by (7). Similarly, we may restrict attention to the Q-functions defined by (14) since the inverse Lorenz family of inequality measures $\{G^*_k : k = 1, 2, ...\}$ uniquely determines the inverse Lorenz curve and consequently the Lorenz curve. These results are summarized in Theorem 7.

THEOREM 7. *Let $L_1$ and $L_2$ be members of $\mathbf{L}$ and let $G_k$ and $G^*_k$ be defined by (8) and (15), respectively. Then the following statements are equivalent,*

*(i)* $L_1(u) \geq L_2(u)$ *for all* $u \in [0,1]$ *and the inequality holds strictly for at least one* $u \in \langle 0,1 \rangle$

*(ii)* $G_k(L_1) < G_k(L_2)$ *for k=1,2,...*

*(iii)* $G^*_k(L_1) < G^*_k(L_2)$ *for k=1,2,...*

Note that the most inequality averse $J_P$-measure is obtained as the preference function approaches

(26)
$$P_a(t) = \begin{cases} 0, & t = 0 \\ 1, & 0 < t \leq 1. \end{cases}$$

As $P_a$ is not differentiable, it is not a member of the family $\mathbf{P}_1$ of inequality averse preference functions, but it is recognizable as the upper limit of inequality aversion for members of $\mathbf{P}_1$.

Inserting (26) in (6) yields

(27)
$$J_{P_a}(L) = 1 - \frac{F^{-1}(0+)}{\mu},$$

where $\mu$ is the mean income and $F^{-1}(0+)$ is the lowest income. Hence, the inequality measure $J_{P_a}$ corresponds to the Rawlsian relative maximin criterion.

By examining the inequality aversion properties of $J_Q^*$ -measures we find that the upper limit of inequality aversion is attained as the preference function approaches

$$Q_a(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t = 1. \end{cases} \tag{28}$$

Inserting (28) in (13) yields

$$J_{Q_a}^*(L) = 1 - \frac{\mu}{F^{-1}(1)}, \tag{29}$$

where $F^{-1}(1)$ is the largest income. Thus, $J_{Q_a}^*$ is the upper limit of inequality aversion for members of $\mathbf{Q}_1$ and is in this respect "dual" to the Rawlsian relative maximin criterion. Thus, we may denote it the relative minimax criterion. In contrast to the Rawlsian relative maximin criterion the relative minimax criterion focuses on the relative income of the best-off unit. If decisions are based on the relative minimax criterion, the income distributions for which the largest relative income is smaller is preferred, regardless of all other differences. The only transfers which decrease inequality are transfers from the richest unit to anyone else.

The two axiomatic based approaches for measuring inequality differ notably with respect to their descriptions of the most inequality averse behavior. When evaluation of inequality is based on the $J_P$-measures, raising the emphasis on transfers occurring lower down in the Lorenz curve attains the most inequality averse behavior. By contrast, if inequality is assessed in terms of $J_Q^*$ -measures, raising the emphasis on transfers occurring higher up in the Lorenz curve attains the most inequality averse behavior. Note that this difference in inequality aversion originates from the difference between Axioms 4 and 5.

## Appendix

### *Proofs*

LEMMA 1. *Let $\gamma$ be a function on $[0,\infty)$ defined by*

$$\gamma(x) = x \log\left(\frac{x+1}{x+2}\right) + \log(x+2) - \log 2.$$

*Then $x = 1$ is a unique root of $\gamma(x) = 0$ in $\langle 0,\infty\rangle$.*

Proof. By straightforward first and second order differentiation of $\gamma(x)$ we get

$$\gamma'(x) = \log\left(\frac{x+1}{x+2}\right) + \frac{x}{x+1} - \frac{x-1}{x+2}$$

and

$$\gamma''(x) = -\frac{x^2 - x - 3}{(x+1)^2 (x+2)^2}.$$

Hence, $\gamma''(x)=0$ if $x = x_1 \equiv \frac{1}{2}\left(1+\sqrt{13}\right)$ and

$$\gamma''(x)\begin{cases} >0 & \text{if} \quad x \in [0, x_1) \\ <0 & \text{if} \quad x \in \langle x_1, \infty\rangle. \end{cases}$$

Thus, $\gamma(x)$ is convex for $x \in [0, x_1)$ and concave for $x \in \langle x_1, \infty\rangle$. Moreover, noting that $\gamma(0) = 0$ and $\gamma(x_1) > 0$, $\gamma(x)$ can at most have one root in $\langle 0, x_1\rangle$ and at most one root in $\langle x_1, \infty\rangle$. However, observing that $\gamma(x) > 0$ for $x > x_1 > 1$, we have that $\gamma(x) > 0$ for all $x \in [x_1, \infty)$ and thus that $x = 1$ is the only root in $\langle 0, \infty\rangle$.

Q.E.D.

LEMMA 2. *Let H be the family of bounded, continuous and non-negative functions on [0,1] which are positive on $\langle 0,1\rangle$ and let g be an arbitrary bounded and continuous function on [0,1]. Then*

$$\int g(t)\, h(t)\, dt > 0 \quad \text{for all} \quad h \in H$$

*implies*

$$g(t) \geq 0 \quad \textit{for all} \quad t \in [0,1]$$

*and the inequality holds strictly for at least one* $t \in \langle 0,1 \rangle$.

The proof of Lemma 2 is known from mathematical textbooks.

**Proof of Theorem 6**. To prove the equivalence of (i) and (ii) we use that $L_1^{-1}(u) \leq L_2^{-1}(u)$ for all $u \in [0,1]$ if and only if $L_1(u) \geq L_2(u)$ for all $u \in [0,1]$. Next, from the definition (11) of $J_Q^*$ and using integration by parts it follows that

$$J_Q^*(L_2) - J_Q^*(L_1) = \int_0^1 Q''(t)\left(L_2^{-1}(t) - L_1^{-1}(t)\right) dt .$$

Hence, if (ii) holds then $J_Q^*(L_2) - J_Q^*(L_1) > 0$ for all $Q \in \mathbf{Q}_1$.

The converse statement follows by straightforward application of Lemma 2.

Q.E.D.

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