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A Model-based Approach to Variance Estimation for Fixed Weights and Chained Price Indices

Li-Chun Zhang[1]

There is presented a model-based approach to variance estimation for both short-term fixed weights and long-term chained price indices. The individual price observations are treated as random variables whose probability distribution in part depends on some unknown index parameters that are postulated as the underlying theoretical inflation rates. The variance estimate summarizes the amount of variation in the potentially noisy price signals, providing an intuitive measure of uncertainty surrounding the estimated average price development.

Key words: Fixed weights index; index parameter; robust variance estimation; chaining.

1. Introduction

The calculation of price indices has been one the most important tasks in economic statistical production for well over a hundred years. Theoretical approaches include the test approach (see Balk l995), the economic approach (see Diewert 1981) and the stochastic approach (see Clements, Izan, and Selvanathan 2006). I refer to International Labour Organization (2004) for a comprehensive survey of the theory and practice concerning the Consumer Price Index (CPI), which often serves as a reference for the construction of many other price indices.

However, there is an “absence of systematic and generally accepted knowledge” (International Labour Organization 2004, Chapter 5, p. 76) about the issue of variance estimation. The CPI manual recommends nevertheless the calculation of sampling variance, and relates in brief the experiences from the US, Sweden, France and Luxembourg. While a sampling-based approach may seem natural given that the price observations are typically collected in sample surveys, some formidable difficulties need to be overcome before the methodology becomes feasible in general. A discussion of the sampling-based approach is given in Section 2.

In this paper I outline a model-based approach to variance estimation for both fixed weights and chained price indices that are being calculated in practice by many statistical agencies. The individual price observations are treated as random variables whose probability distribution, depends in part on some theoretical *index parameters* that are postulated as the underlying price development (or inflation rate) of interest. A calculated index is considered as an estimate of the corresponding index parameter, based on a number of disparate movements of individual prices that have been observed. The associated variance estimate summarizes the amount of variation in these potentially noisy price signals, thereby providing an intuitive measure of the uncertainty surrounding

[1] Statistics Norway. Kongensgate 6, PB 8131 Dep, N-0033 Oslo, Norway. Email: lcz@ssb.no

the estimated underlying inflation rate. Fixed weights indices are treated in Section 3, and chained indices in Section 4. A numerical illustration is given in Section 5. Section 6 contains a final unifying remark.

2. On the Sampling-based Approach

For a sampling-based approach to variance estimation, one must first define the target parameter of estimation as a finite-population characteristic. In the price index context, this is set up in the recent CPI manual (International Labour Organization 2004, Chapter 5, page 69) as follows: (I) "a *universe* consisting of a finite population of units (e.g., products)", (II) "one or more *variables* that are defined for each unit in the universe (e.g., price and quantity)", and (III) "a formula which combines the values of one or more of these variables for all units in the universe into a single value called a *parameter* (e.g., the Laspeyres index)".

A few observations are worth noting. Firstly, if a price index is meant to summarize the changes in prices between two different time periods, then there is a *time* dimension attached to both the units in (I) and the associated variables in (II). The CPI, for instance, is often calculated on a month-to-month basis. But the price of a product is not necessarily the same throughout a month. Secondly, there is a *geographical* and/or *outlet* dimension, in the sense that the price of the same product may vary from one outlet to another as well as from one region in the country to another. Thirdly, there is also a *package* dimension, in the sense that the price of a product may vary according to the quantity of a particular transaction. This is for instance the case when the same beverage is sold at different prices per litre depending on the package size. This is also the case if a lower price is offered to the customer who purchases the same product in large quantities or different products in a bundle.

Thus, basically, the target index parameter, pertaining any two time periods of comparison, will have to be defined on the basis of *all* the transactions that have taken place, because in principle a unit (e.g., a product) and its price may be specific to time, location, or a particular transaction. Now, it can be argued that it is possible to average out a certain dimension, and thereby reducing the size of the finite population, by appropriate aggregation and the use of a corresponding *unit value* price. For instance, one can aggregate the sales of the same product at a particular outlet over an entire period of time (e.g., a month) and derive the unit value price as the ratio between the total sales revenue and total quantity. However, such an approach would require the knowledge of price and quantity in all the involved transactions, such that the parameter of the reduced universe will still have to be determined on the basis of the same amount of information.

In practice, a sample of price observations usually have an outlet dimension and a product dimension. Often it is possible to select the outlets according to a sampling design. But probability sampling of products is rare. Use of centrally selected representative products or self-nomination by the respondents is much more common. A model of sampling needs to be postulated in such cases (e.g., Dalén and Ohlsson 1995), and the calculated variance will then acquire a *quasi-sampling* nature. Moreover, the geographical and/or outlet dimension is generally not adequate on its own: also the time and package dimensions should be taken into consideration when selecting the products. One may adopt a reduced product universe by assuming appropriate aggregation over the time and package dimensions. But the prices that are collected on particular occasions will then contain measurement errors with regard to the target unit value prices. And the errors are inestimable based on the information available, such that the approach will damage the relevance of the calculated sampling variance as a measure of uncertainty. In summary, the current practice of price data collection generally does not permit a purely sampling-based approach to variance estimation. Also a quasi-sampling approach will be difficult, if the modelling of the sampling distribution must take into account the time and package dimensions.

In recent years, complete transaction records such as scanner data have been made available in some areas (or branches) of price indices. Data collection then takes the form of a census instead of a sample survey, and sampling variance is eliminated in principle. However, the price index being an average of a number of disparate movements of individual prices, it seems nevertheless sensible to ask how strong the trend is that exhibits itself in a single index number, or how clear the evidence is behind the average price development. This brings us to the interpretation of the variance being calculated. Consider the following two examples.

Example One. Suppose census of the CPI universe, which under the sampling framework would give us the exact index parameter with zero variance. Now, imagine that there had been sold one *extra* pair of shoes in the universe, which would have given rise to a different index parameter. While numerically the difference between the two may be negligible, conceptually the target parameter must be different in the two situations under the sampling-based framework. But the underlying economic conditions can hardly be regarded as different with or without that extra pair of shoes. So the question is, can the index parameter be denned in such a way that it remains the same in these two cases, while allowing the numerical difference between the two calculated indices to be the difference between two estimates?

Example Two. Suppose census of a universe of three commodities, whose prices in the reference period are given as (5,2,3). Consider 3 alternative scenarios for the corresponding prices in the current period: (i) (6,3,3), (ii) (4,4,4), and (iii) (6,2.4,3.6). The ratio between the average prices in the current and reference periods is 1.2 in all the 3 scenarios. Because of census the sampling variance is always zero. But is the trend equally strong in all the three cases? Is there no way in which one might e.g., find the message to be particularly clear in case (iii)? Can we find a variance measure that is zero only if the prices are observed to move in perfect unison?

With these questions in mind we now move on to a model-based approach to variance estimation.

3. Variance Estimation for Fixed Weights Price Indices

In practice fixed weights price indices are calculated in two stages. First, the universe of products (or services) is divided into a number of mutually exclusive groups, and *elementary* indices are calculated for each of these groups between the current period and a price reference period. Apart from some special situations such as when scanner data are available, the elementary indices are calculated on the basis of a sample of matched price observations only, without knowledge of the quantity or revenue of the products.

Next, the elementary indices are averaged to obtain higher-level indices, using their relative values (i.e., *value share)* as the weights. In particular, the weights may be constructed on the basis of surveys from previous years, such that they may refer to a different time period than the price reference period. As time moves on, a series of price indices may be calculated in this way, using the same weights and reference prices. It is important to emphasize that, while these are merely referred to as the fixed weights indices, also the reference prices are in fact being held fixed all the time. Changes in either the reference prices or the weights necessarily calls for chaining of indices, which will be dealt with in Section 4.

3.1. Formal Expression of Fixed Weights Price Indices

Reference period. Denote by s the *price reference* period, and by t the *statistical* (or current) period. Denote by b the *weights* period. Typically, not only does b differ from s, they can also have different durations. For example, b may refer to a year, while s (and t) may refer to a month.

Aggregation. *Aggregations* are defined according to one or more classification variables. At the lowest level we have the *elementary* aggregations, or elementary groups, denoted by $i = 1, \ldots, M$. A higher level aggregation consists of a set of elementary groups. No higher level aggregation may cut across an elementary one. Often the aggregations are organized in a hierarchical structure, with disjoint aggregations on the same level of hierarchy, and each of them consists of one or more aggregations at the level immediately below. Denote by G an arbitrary aggregation. An elementary aggregation is denoted by $G = \{i\}$, whilst the *total* aggregation is denoted by U.

Price relative. Denote by (ij) the j-th product in the i-th elementary group. Denote by p_{ij}^{t} its price in the statistical period, and denote by p_{ij}^{s} its price in the price reference period. Let $I_{ij}^{s,t} = p_{ij}^{t}/p_{ij}^{s}$ be the corresponding *price relative* from s to t. Notice that the classification of the products and the associated pricing method can often vary from one elementary group to another depending on the data available. For instance, unit value price may be available for narrowly denned products in certain elementary groups, but not the others. The basic requirement here is that the products (and prices) can be matched (or spaired) over the periods s and t.

Elementary index. Denote by $P_{i}^{s,t}$ the i-th *elementary index*. We consider here only elementary indices that are calculated *without* the weights. This is the first-step in the calculation of fixed weights index. Let n_i be the number of prices relatives. The three most common elementary indices are: (i) the *Carli* index (Carli 1804), or the arithmetic mean of price relatives, given by

$$P_{i}^{s,t} = \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{p_{i,j}^{t}}{p_{ij}^{s}} = \frac{1}{n_i} \sum_{j=1}^{n_i} I_{ij}^{s,t} \tag{1}$$

(ii) the *Dutot* index (Dutot 1738), or the ratio between price averages, given by

$$P_{i}^{s,t} = \frac{\sum_{j=1}^{n_i} p_{ij}^{t}/n_i}{\sum_{j=1}^{n_i} p_{ij}^{s}/n_i} = \frac{\sum_{j=1}^{n_i} p_{ij}^{t}}{\sum_{j=1}^{n_i} p_{ij}^{s}} \tag{2}$$

and (iii) the *Jevons* index (Jevons 1863), or the geometric mean of price relatives, given by

$$p_i^{s,t} = \left(\prod_{j=1}^{n_i} \frac{p_{ij}^t}{p_{ij}^s} \right)^{1/n_i} = \exp\left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} \log p_{ij}^t - \frac{1}{n_i} \sum_{j=1}^{n_i} \log p_{ij}^s \right\} \tag{3}$$

Notice that we have $P_i^{s,t} \equiv I_{i1}^{s,t} = p_{i1}^t / p_{i1}^s$, or simply $p_i^{s,t} = I_i^{s,t} = p_i^t / p_i^s$, if and only if $n_i = 1$, i.e., if the elementary group of concern consists of only a single product.

Weight. A *total* index is a weighted average of all the elementary indices. The weights are denoted by w_i^b, for $i = 1, \ldots, M$. Formally the weights must be positive and sum to unity, i.e., $w_i^b > 0$ and $\sum_{i=1}^m w_i^b = 1$. Typically, w_i^b stands for the value share of the i-th elementary aggregation, derived from the information collected in the weights period b. In addition, *partial* indices are customarily calculated for higher level aggregations. The corresponding weights for the partial index of aggregation G, denoted by $w_{i(G)}^b$ where $\sum_{i \in G} w_{i(G)}^b = 1$, are given by

$$w_{i(G)}^b = w_{i(U)}^b / \sum_{k \in G} w_{k(U)}^b = w_i^b / \sum_{k \in G} w_k^b$$

However, for simplicity we may use w_i^b and $w_{i(G)}^b$ interchangeably where the context is clear.

Higher level index. An *L-type* index for period t with reference periods s and b is given by

$$P^{s,t}(b) = \sum_i w_i^b p_i^{s,t} \quad \text{for} \quad \sum_i w_i^b = 1 \tag{4}$$

with the summation being over all the elementary aggregations involved. For simplicity we do not include in the above notation the aggregation for which the index is defined. But we write $P_G^{s,t}(b) = \sum_{i \in G} w_{i(G)}^b P_i^{s,t}$ where the emphasis is necessary. Notice that an L-type index $P^{s,t}(b)$ becomes a Young index (Young 1812) if $n_i = 1$ for all $i = 1, 2, \ldots, M$, and if $w_i^b = v_i^b = q_i^b p_i^b$ where q_i^b is the quantity of the i-th product in period b. It becomes a Laspeyres index (Laspeyres 1871) if *in addition* $b = s$ since, then, $P^{s,t}(s) = \sum_i \left\{ \left(q_i^s p_i^s \right) / \left(\sum_k q_k^s p_k^s \right) \right\} \left(p_i^t / p_i^s \right) = \left(\sum_i q_i^s p_i^t \right) / \left(\sum_i q_i^s p_i^s \right)$. It is thus clear that it would be misleading to refer to the index (4) either as a Laspeyres or a Young index in general. Nevertheless, the association is there. Hence the term L-type index.

Next, a *P-type* index for statistical period t with reference periods s and b is given by

$$\wp^{s,t}(b) = \left\{ \sum_i w_i^b \left(P_i^{s,t} \right)^{-1} \right\}^{-1} \tag{5}$$

It becomes a Paasche index (Paasche 1874) if $n_i = 1$ and $w_i^b = v_i^b$ and $b = t$ since, then,

$$\wp^{s,t}(t) = \left\{ \sum_i \frac{q_i^t p_i^t}{\sum_k q_k^t p_k^t} \left(\frac{p_i^t}{p_i^s} \right)^{-1} \right\}^{-1} = \left\{ \frac{\sum_i q_i^t p_i^s}{\sum_i q_i^t p_i^t} \right\}^{-1} = \frac{\sum_i q_i^t p_i^t}{\sum_i q_i^t p_i^s}$$

Hence the term P-type index. The *P*- and *L*-type indices differ only in the way of averaging: while an *L*-type index is a weighted arithmetic mean of the elementary indices, a *P*-type index is a weighted harmonic mean of the same elementary indices. An elementary index is said to satisfy the time reversal test (Fisher 1922) provided that $P_i^{s,t} = 1/P_i^{t,s}$. Given this is the case, we have

$$\wp^{s,t}(b) = \left\{ \sum_i w_i^b \left(P_i^{s,t}\right)^{-1} \right\}^{-1} = \left\{ \sum_i w_i^b P_i^{t,s} \right\}^{-1} = 1/P^{t,s}(b)$$

Both the Dutot and Jevons indices satisfy the time reversal test, but not the Carli index. Finally, the geometric mean of the L- and P-type indices, i.e., $\sqrt{P^{s,t}(b)\wp^{s,t}(b)}$, will be referred to as an *F-type* index due to the obvious analogy to the Fisher index (Fisher 1922).

3.2. *Models for Elementary Indices*

Each of the three elementary indices, defined in (1) to (3), can be motivated by the best linear unbiased estimator (BLUE) of the regression coefficient in a corresponding linear regression model. The models for the Carli and Jevons indices are often mentioned in discussions of the unweighted stochastic approach (e.g., International Labour Organization 2004, Paragraphs 16.74–16.75).

Carli index. The model for the Carli index is given by

$$I_{ij}^{s,t} = \frac{p_{ij}^t}{p_{ij}^s} = \theta_i + \varepsilon_{ij} \quad \text{where } E(\varepsilon_{ij}) = 0 \quad \text{and} \quad V(\varepsilon_{ij}) = \sigma_i^2 \quad \text{and} \quad Cov(\varepsilon_{ij}, \varepsilon_{ik}) = 0 \qquad (6)$$

i.e., a group homogeneity model for price relatives, with both constant group mean and variance. The Carli index is the BLUE of θ_i, which can be referred to as the *elementary index parameter*. Notice that the full notation should be $\theta_i^{s,t}$, which will be necessary when we come to the chained index in Section 4. But for convenience we will use θ_i where the context is clear. Notice also that the above model can be rewritten as

$$p_{ij}^t = \theta_i p_{ij}^s + \varepsilon_{ij} \quad \text{where } E(\varepsilon_{ij}) = 0 \quad \text{and} \quad V(\varepsilon_{ij}) = \sigma_i^2 \left(p_{ij}^s\right)^2 \quad \text{and} \quad Cov(\varepsilon_{ij}, \varepsilon_{ik}) = 0$$

i.e., a group ratio model for individual prices p_{ij}^t given p_{ij}^s where the residual has a variance that is proportional to $\left(p_{ij}^s\right)^2$. Under model (6) the theoretical variance of the Carli index is given by

$$V\left(P_i^{s,t}\right) = n_i^{-2} \sum_j V\left(I_{ij}^{s,t}\right) = \sigma_i^2 / n_i \qquad (7)$$

Dutot index. The model for the Dutot index is given by

$$p_{ij}^t = \theta_i p_{ij}^s + \varepsilon_{ij} \quad \text{where } E(\varepsilon_{ij}) = 0 \quad \text{and} \quad V(\varepsilon_{ij}) = \sigma_i^2 p_{ij}^s \quad \text{and} \quad Cov(\varepsilon_{ij}, \varepsilon_{ik}) = 0 \qquad (8)$$

i.e., a group ratio model for individual prices with residual variance proportional to p_{ij}^s. Thus, the only difference from model (6) is the variance assumption. The Dutot index is the BLUE of the elementary index parameter θ_i in (8), with a theoretical variance given by

$$V\left(P_i^{s,t}\right) = \left(\sum_{j=1}^{n_i} p_{ij}^s\right)^{-2} \left\{\sum_j V\left(p_{ij}^t\right)\right\} = \sigma_i^2 / \left(\sum_{j=1}^{n_i} p_{ij}^s\right) \tag{9}$$

Jevons index. The model for the Jevons index is given by

$$\log I_{ij}^{s,t} = \mu_i + \varepsilon_{ij} \quad \text{where } E(\varepsilon_{ij}) = 0 \quad \text{and} \quad V(\varepsilon_{ij}) = \sigma_i^2 \quad \text{and} \quad Cov(\varepsilon_{ij,} \varepsilon_{ik}) = 0 \tag{10}$$

i.e., a group homogeneity model for the logarithm of price relatives. It can be rewritten as

$$p_{ij}^t = p_{ij}^s \exp(\mu_i) \exp(\varepsilon_{ij})$$

i.e., a non-linear regression model for p_{ij}^t given p_{ij}^s with multiplicative random errors. Notice that, while the logarithm of the Jevons index is the BLUE of μ_i, the Jevons index itself is not the BLUE but a plug-in estimator of the elementary index parameter $\theta_i = \exp(\mu_i)$. Under model (10) its approximate theoretical variance is given by

$$V\left(P_i^{s,t}\right) \approx e^{2\mu_i} n_i^{-2} \sum_j V\left(\log I_{ij}^{s,t}\right) = e^{2\mu_i} \sigma_i^2 / n_i = \theta_i^2 \sigma_i^2 / n_i \tag{11}$$

We note that the above model-based variances of the elementary indices have the following properties: (1) The target parameter is of a theoretical nature and does not depend on the number of price observations available. It is a characterization of the underlying economic conditions, rather than a direct statistic of the actual transactions that have taken place. (2) The variance of each elementary index depends on a within-group variance component σ_i^2, which provides motivation that in practice one should strive to achieve homogeneous elementary groups that have a low dispersion of individual price movements. Indeed, zero variance is the case only if all the prices move in perfect unison. (3) The variance is inversely related to the number of observed price relatives. This confirms to the intuition that, given the within-group variance component σ_i^2, the estimation uncertainty is reduced as the number of observations increases.

It is worth pointing out a subtlety in our use of statistical models above. In the literature of the so-called stochastic approach to index number theory, a statistical model may be introduced to *motivate* a particular index formula. For instance, we arrive at the Carli index if we assume constant variance $V(\varepsilon_{ij}) = \sigma_i^2$ in model (6). Whereas changing it to $V(\varepsilon_{ij}) = \sigma_i^2 / v_{ij}^s$ where $v_{ij}^s = q_{ij}^s p_{ij}^s$ is the transaction value of product (ij) in period s, we would obtain the BLUE of θ_i given instead by $\left(\sum_j q_{ij}^s p_{ij}^t\right) / \left(\sum_j q_{ij}^s p_{ij}^s\right)$, i.e., a Laspeyres index. Now, it is *not* my intention to motivate the use of a particular elementary index through the underlying statistical model, regardless of whether such a model is appropriate in a given situation. Such a decision is left to the measurement economist. Rather, we take as our starting point that a particular elementary index formula has been chosen, and use the corresponding statistical model as a guidance for how to summarize the variation in the price observations. For this purpose a statistical model is considered to correspond to a particular index formula on two accounts: (a) the index is an unbiased (or approximately unbiased) estimator of a parameter of the model, i.e., θ_i in the above, and (b) the index is apparently an efficient estimator under the same model, i.e., the BLUE characterization. For instance, it does not seem natural to summarize the uncertainty surrounding a Dutot index by expression (7), because the Dutot index formula does not have nice properties under model (6) which gives rise to (7). Instead, expression (9) seems sensible because the Dutot index has nice properties under Model (8).

3.3. *Higher Level Indices*

Higher level indices are weighted averages of the elementary indices. I propose to evaluate the variance of the higher level indices conditional on the actual weights. Typically, as in the case of the CPI, the weights are derived from other sources that are independent of the price observations. It is thus in principle straightforward to incorporate the variance of weights estimation through a Taylor linearization technique. Such an unconditional variance can be calculated from time to time in order to assess the relative magnitude of the variance components due to the weights and the price observations. On a running basis, however, a series of fixed weights indices are all based on the same weights, such that the conditional variances seem more appropriate for comparison of the fixed weights indices of different statistical periods.

Now, conditional on the weights, the variance of a higher level index can be decomposed into the variances at the elementary level. For an L-type index, we have

$$V(P^{s,t}(b)) = \sum_i a_i V\left(P_i^{s,t}\right) \text{ where } a_i = \left(w_i^b\right)^2 \tag{12}$$

Whereas by means of Taylor linearization the variance of a P-type index is given by

$$V(\wp^{s,t}(b)) \approx \sum_i a_i V\left(P_i^{s,t}\right) \text{ where } a_i = \left(w_i^b\right)^2 E(\wp^{s,t}(b))^4 / E\left(P_i^{s,t}\right)^4 \tag{13}$$

It is important to clarify the target higher-level index parameter. The index parameter that corresponds to an L-type index is given by

$$\theta = \theta^{s,t}(b) = \sum_i w_i^b \theta_i^{s,t} \tag{14}$$

We can regard $\{w_i^b\}$ as a probability mass function since $\sum_i w_i^b = 1$ and $w_i^b > 0$. Suppose one were to take a measurement of the price change between s and t by the following two-stage procedure: first, select an elementary aggregation by the probability w_i^b; second, select a product *randomly* within the elementary group selected at the first stage. The parameter θ is then the *expected* price relative that one would observe by such a procedure. Moreover, the index parameter that corresponds to a partial L-type index is the *conditional* expectation of the observed price relative given that the observation is to be taken within the aggregation G, and is given by

$$\theta_G = \theta_G^{s,t}(b) = \sum_{i \in G} w_{i(G)}^b \theta_i^{s,t} \text{ where } w_{i(G)}^b = w_i^b / \sum_{i \in G} w_i^b$$

Finally, a P-type type index can be considered to aim at $\theta = \left(\sum_i w_i^b / \theta_i^{s,t}\right)^{-1}$.

Theil (1967) provides an interpretation of the target index parameter for the Törnqvist-Theil index as the expected logarithmic price relatives. The index parameter (14) extends Theil's approach in three respects First, the M individual products are generalized to M product groups, and the situation of $n_i \equiv 1$ for all i's can be treated as a special case. Second, there is now an explicit distinction between the price relatives as random variables and θ as the unknown theoretical parameter. A computed index $P^{s,t}(b)$ is always an estimate, but never the parameter itself. Third, the index parameter (14) is not restricted to a particular index formula, such as the Törnqvist-Theil index. Different elementary indices can be accommodated under different statistical models. The approach to variance estimation remains otherwise the same.

3.4. *Robust Variance Estimation*

A direct plug-in variance estimator can be obtained by replacing (σ_i^2, θ_i) with $\left(\hat{\sigma}_i^2, P_i^{s,t}\right)$ wherever they appear in the variance formulae (7), (9) and (11). Variance estimates for higher level indices can then be obtained through (12) and (13), where we replace $E(\wp^{s,t}(b))$ with $\wp^{s,t}(b)$ in (13). But this is *not* a method I prefer. Take for example the Carli index. Although the assumption of constant variance in the model (6) helps to motivate the Carli index as the BLUE of θ_i, there is no guarantee that the assumption is appropriate. Diewert (1995) raised a similar criticism of the stochastic approach of Clements and Izan (1987). We can handle the problem by using a variance estimation technique that only requires the residuals to have zero mean. It is unnecessary to impose any variance assumption, and each residual is allowed to have its individual variance.

The idea of such a robust variance estimation approach has a long history in the economic statistical literature. See Chapter 5 in Valliant, Dorfman, and Royall (2000) for a description under the general linear model. The details of the three elementary models are now given below.

Carli index. The first expression in the Formula (7) suggests that we need to estimate each individual variance, denoted by $\sigma_{ij}^2 = V\left(I_{ij}^{s,t}\right)$. The second expression assumes that $\sigma_{ij}^2 = \sigma_i^2$. Let

$$e_{ij} = I_{ij}^{s,t} - \hat{\theta}_i = I_{ij}^{s,t} - P_i^{s,t} \text{ where } E\left(e_{ij}^2\right) = \left(1 - \frac{1}{n_i}\right)^2 \sigma_{ij}^2 + \frac{1}{n_i^2} \sum_{k \neq j} \sigma_{ik}^2$$

We now estimate the variance of the Carli index by

$$v_i = \hat{V}\left(P_i^{s,t}\right) = n_i^{-2} \frac{n_i}{n_i - 1} \sum_j e_{ij}^2 = \frac{\sum_j e_{ij}^2}{n_i(n_i - 1)} \tag{15}$$

It is straightforward to verify that v_i is unbiased provided $E(\varepsilon_{ij}) = 0$ whether or not $\sigma_{ij}^2 = \sigma_i^2$.

Dutot index. Again, in general we need to estimate $\sigma_{ij}^2 = V\left(p_{ij}^t\right)$. Let

$$e_{ij} = p_{ij}^t - \hat{\theta}_i p_{ij}^s = p_{ij}^t - P_i^{s,t} p_{ij}^s = p_{ij}^t - \frac{\sum_k p_{ik}^t}{\sum_k p_{ik}^s} p_{ij}^s = \left(1 - \frac{p_{ij}^s}{\sum_k p_{ik}^s}\right) p_{ij}^t + \frac{p_{ij}^s}{\sum_k p_{ik}^s} \sum_{l \neq j} p_{il}^t$$

I now estimate the variance of the Dutot index by

$$v_i = \hat{V}\left(P_i^{s,t}\right) = \left(\sum_k p_{ik}^s\right)^{-2} \sum_j \left(1 - \frac{p_{ij}^s}{\sum_k p_{ik}^s}\right)^{-1} e_{ij}^2 \tag{16}$$

which is approximately unbiased provided $E(\varepsilon_{ij}) = 0$ irrespective of the assumption $\sigma_{ij}^2 = \sigma_i^2 p_{ij}^s$.

Jevons index. In general we need to estimate $\sigma_{ij}^2 = V\left(\log I_{ij}^{s,t}\right)$. Let

$$e_{ij} = \log I_{ij}^{s,t} - \hat{\mu}_i = \log I_{ij}^{s,t} - \log P_i^{s,t}$$

Robust variance estimation on this log-scale is similar to the case of the Carli index above. An approximate unbiased variance estimator of the Jevons index is then given by

$$v_i = \hat{V}\left(P_i^{s,t}\right) = \left(P_i^{s,t}\right)^2 \frac{\sum_j e_{ij}^2}{n_i(n_i - 1)} \tag{17}$$

3.5. *More Complex Models*

Under the models (6), (8) and (10) we assume that the residuals ε_{ij} of individual prices are independent of each other *conditional on* the within-group parameters θ_i. Generally, however, clustering among products (or commodities) implies that the residuals may not be independent *unconditionally*. It is possible to allow for such unconditional variance-covariance structure under multilevel modelling, by introducing additional random effects or intra-cluster correlations.

For example, Clements and Izan (1987) allow for commodity-specific parameters which are interpreted as the systematic part of the change in the price relatives. We can adapt their approach to the current setting and, say, extend the model (10) for the Jevons index as follows. Put

$$\log p_{ij}^t - \log p_{ij}^s = \mu + \beta_i + \varepsilon_{ij} \quad \text{where}$$

$$E(\varepsilon_{ij}) = 0 \text{ and } V(\varepsilon_{ij}) = \sigma_i^2 \text{ and } Cov(\varepsilon_{ij}, \varepsilon_{ik}) = 0$$

Suppose we regard the group-specific parameter β_i as a random variable itself, with zero mean and variance, say, σ_β^2. Then, the residuals $u_{ij} = \log p_{ij}^t - \log p_{ij}^s - \mu = \beta_i + \varepsilon_{ij}$ are conditionally independent given the β's, but unconditionally dependent due to the fact that u_{ij} and u_{ik} share a common random effect β_i for $j \neq k$. Under a more complex model, Valliant (1992) allows for intra-cluster correlation among items from the same establishment, in addition to a first-order autoregressive correlation over time among the establishments from the same stratum.

Such multilevel or time-series models would lead to different assessment of the estimation uncertainty. Even more importantly, they imply that price indices can be calculated in a quite different way than the current practice, because the information of, say, $\mu_i = \mu + \beta_i$ is no longer isolated in the i-th elementary group. But the issue is beyond the scope of this paper. Thus we shall restrict myselves to the three elementary models above, and adopt the view that the variance is evaluated *conditional on* the potential extra random effects that may be introduced.

4. Variance Estimation for Chained Price Indices

4.1. Preliminary

Sooner or later one or both of the reference periods s and b will have to be changed. There arises thus a need for *chaining* subsequent series of indices. We consider here in detail only the chaining of *two* fixed weights indices, from which the general situation can be inferred inductively. Numerically, chaining amounts to a multiplicative adjustment of the indices. There is thus some similarity to another multiplicative operation called *normalization:* for presentation an index series must be set to unity at a certain point, which is to be referred to as the *index reference period*.

Normalization is always achieved by means of *separate* re-scaling for each aggregation G of interest. However, while normalization should never lead to a *break*, i.e., change in the index development, breaks may be unavoidable or even desirable as a result of chaining. One should therefore maintain the conceptual distinction between chaining and normalization. Indeed, I recommend that one only keep record of a single chained index series, and carry out the normalization only whenever it is necessary for presentation.

In terms of notation I shall denote by $\tilde{P}^t_G$ a *chained* L-type index of aggregation G. Similarly $\tilde{\wp}^T_G$ denotes a chained P-type index. Not all the reference periods can be uniquely settled in a chained index series. They will be included in the notation when it is desirable as well as possible. Thus, e.g., $\tilde{P}^t(b)$ stands for a chained index based on the same set of weights from period b.

Denote by d the *chaining* point at which the two series of indices are joined together. As a rule in practice it is normally required that chaining should not affect the development in the indices that have already been published. We shall therefore assume in the sequels that d is the period of the last published index before chaining, since otherwise one would have to delay the publication for some time. Thus, let $P^{s,t}_G(b)$ be the fixed weights index before chaining, we require that

$$\tilde{P}^{t_2}_G / \tilde{P}^{t_1}_G = P^{s,t_2}_G(b) / P^{s,t_1}_G(b) \;\text{ for }\; t_1 < t_2 \leq d \tag{18}$$

Notice that the condition (18) does not imply that price indices are never revised. The calculation of the "best" index series is sometimes carried out retrospectively as the required information becomes available. But this is quite another issue than routine real-time chaining operations.

An important and convenient consequence of the condition (18) is that variance estimation can be made conditional on all the calculated indices up to the point d, i.e., $\{\tilde{P}_G^t; t \leq d\}$ for all the aggregations, to be referred to as the *trunk* of the chained index. What is variable is the *head* of the chained index, i.e., $\{\tilde{P}_G^t; t > d\}$ for all G. Without conditioning, a chained index would almost surely become more and more uncertain as time went on. While this may be true unconditionally, it is not very helpful for the comparison of indices calculated at different time points. The conditional variance seems more appropriate for such purposes.

4.2. *Chained Elementary Index*

Since an elementary index is calculated without the weights, only a change in the price reference period s can cause chaining of elementary indices. A common situation arises when direct comparisons with the reference prices become impossible, because old outlets and/or products are replaced by new ones for which the prices from the period s are either unavailable or nonexistent.

Formally, we assume that s is to be "updated" to s', i.e., $s < s'$, and $d = s'$. Put

$$\tilde{P}_i^t = \begin{cases} P_i^{s,t} & \text{for } t \leq d \\ P_i^{s,d} P_i^{d,t} & \text{for } t > d \end{cases}$$

which satisfies Condition (18). Moreover, an elementary index $P_i^{s,t}$ is said to be *transitive* provided $P_i^{s,t} = P_i^{s,s'} P_i^{s',t}$ for arbitrary $s < s' < t$. Both the Dutot and Jevons indices are transitive, but not the Carli index. It is clear that a transitive elementary index does not result in a break, such that it allows one to smoothly update outlets and/or products.

The statistical model for each elementary index retains the same form as before, but has different parameters and residuals for the trunk and the head.

- We may rewrite the Carli index as $P_i^{s,t} = \theta_i^{s,t} + u_i^t$ where $u_i^t = \sum_{j=1}^{n_i} \varepsilon_{ij}^t / n_i$, such that

$$E\left(\tilde{P}_i^t\right) = E\left(P_i^{s,d} P_i^{d,t}\right) = E\left\{\left(\theta_i^{s,d} + u_i^d\right)\left(\theta_i^{d,t} + u_i^t\right)\right\}$$
$$= E\left(\theta_i^{s,d}\theta_i^{d,t} + \theta_i^{s,d}u_i^t + \theta_i^{d,t}u_i^d + u_i^d u_i^t\right)$$
$$= \theta_i^{s,d}\theta_i^{d,t} + 0 + 0 + E\{u_i^d E[u_i^t | (\varepsilon_{i1}^d, \ldots, \varepsilon_{n1}^d)]\} = \theta_i^{s,d}\theta_i^{d,t}$$

  because $E[u_i^t | (\varepsilon_{i1}^d, \ldots, \varepsilon_{in_i}^d)] = E[u_i^t | (p_{i1}^d, \ldots, p_{in_i}^d)] = 0$. Notice that the conditional argument for $E(u_i^d u_i^t)$ is necessary because $p_i^{s,d}$ and $p_i^{d,t}$ are dependent on each other *unconditionally*.
- Similarly, we may rewrite the Dutot index as $P_i^{s,t} = \theta_i^{s,t} + u_i^t$ where $u_i^t = \sum_{j=1}^{n_i} \varepsilon_{ij}^t / \sum_{j=1}^{n_i} p_{ij}^s$, and obtain $E\left(\tilde{P}_i^t\right) = E\left(P_i^{s,d} P_i^{d,t}\right) = \theta_i^{s,d}\theta_i^{d,t}$ by the same argument as above.

- Finally, for the Jevons index, we have $\log P_i^{s,t} = \mu_i^{s,t} + u_i^t$ where $u_i^t = \sum_{j=1}^{n_i} \varepsilon_{ij}^t / n_i$, and $E(\log \tilde{P}_i^t) = E(\log P_i^{s,d} + \log P_i^{d,t}) = \mu_i^{s,d} + \mu_i^{d,t}$, so that $\tilde{P}_i^t$ can be considered to be aiming at exp $(\mu_i^{s,d} + \mu_i^{d,t}) = \theta_i^{s,d}\theta_i^{d,t}$.

In summary, the chained elementary index parameters can be given by a common formula

$$\tilde{\theta}_i^{s,t} = \theta_i^{s,d}\theta_i^{d,t} \quad \text{for } s < d < t \tag{19}$$

For a transitive elementary index we have $\theta_i^{s,t} = \tilde{\theta}_i^{s,t}$, i.e., no break by chaining, and the target index parameter is the same for the chained elementary index and the direct elementary index. Finally, the conditional variance is given by

$$V\left(\tilde{P}_i^t \middle| \tilde{P}_i^d\right) = \left(\tilde{P}_i^d\right)^2 V\left(P_i^{d,t} \middle| \mathbf{p}^d\right) \text{ for } \mathbf{p}^d = \left\{P_{ij}^d; i = 1, \ldots, M \text{ and } j = 1, \ldots, n_i\right\}$$

and the robust variance estimators (15)–(17) can be used to estimate $V\left(P_i^{d,t} \middle| \mathbf{p}^d\right)$ as before.

4.3. *Higher Level Chained Indices*

4.3.1. A General Formula

Higher level chained indices are more complicated in the sense that it is generally not possible to find a set of "weights" such that all higher level chained indices can be expressed as a weighted average of the chained elementary indices. Chaining must be carried out separately for each aggregation of interest, a fact that we emphasize through the use of subscript G in the notation. Let $P_G^{s,d}(b)$ be the fixed weights index up to d, and let $\tilde{P}_G^{d,t}$ be the development from d to t, i.e., the index head. By default the following chaining operation satisfies condition (18), i.e.,

$$\tilde{P}_G^t = P_G^{s,d}(b)\tilde{P}_G^{d,t} \tag{20}$$

Fortunately, it turns out that the common choices of chaining operations in practice can all be expressed in terms of the following general formula for the index head $\tilde{P}_G^{d,t}$ given by

$$\tilde{P}_G^{d,t} = \sum_{i \in G} \tilde{w}_{i(G)} P_i^{d,t} \quad \text{where} \quad \tilde{w}_{i(G)} = \frac{\tilde{w}_i}{\sum_{k \in G} \tilde{w}_i} \quad \text{and} \quad \tilde{w}_i = \frac{w_i^{b'} \Delta_i}{\sum_{k \in U} w_k^{b'} \Delta_k} \tag{21}$$

Here, $P_i^{d,t}$ is an elementary index from d to t, and $w_i^{b'}$ is a genuine set of weights from period b', and $\{\Delta_i; i \in U\}$ is a set of *standardizing factors*, and $\tilde{w}_i$ is a set of *re-standardized* weights. Index (21) is a *re-standardized* L-type index. A *re-standardized* P-type index is given by

$$\tilde{\wp}_G^{d,t} = \left(\sum_{i \in G} \tilde{w}_{i(G)} / P_i^{d,t}\right)^{-1}$$

Notice that, while the weights $\{w_i^{b'}; i \in U\}$ are genuine in the sense that they are calculated on the basis of the value shares in period b', the re-standardized weights may use information from other periods. For example, $\Delta_i = P_i^{t_1,t_2}$, for some time points $t_1 < t_2$, can be used to *price-update* $w_i^{b'}$ from period b' to $(b' + (t_2 - t_1))$ But these are not the genuine weights from period $b' + (t_2 - t_1)$, which would have been denoted by $w_i^{b'+t_2-t_1}$. Notice also that the transformation from $\tilde{w}_i$ to $\tilde{w}_{i(G)}$ is the standard one between the weights for U and those for an arbitrary aggregation G, and has nothing to do with the chaining.

Now, in the calculation of the conditional variance of a chained index $\tilde{P}^t$, *all* the previously calculated indices that belong to the trunk are treated as fixed. So are the restandardized weights for the index head (21), provided the standardizing factors are based on information from the trunk. The variances of $\tilde{P}^{d,t}$ and $\tilde{\wp}^{d,t}$ are thus given by (12) and (13), respectively, with w_i^b replaced by $\tilde{w}_i$. Together formulae (20) and (21) form the basis of variance estimation for higher level chained indices. Below I describe Expression (21) for the common chaining situations.

4.3.2. Change in Price Reference Period

Consider changing the price reference period from s to s', for $s < s'$, while the weights remain fixed. The chaining point will be set at $d = s'$. The first option is simply to calculate the head as a fixed weights index with the same weights and the new reference prices:

$$\tilde{P}_G^t(b) = \begin{cases} P_G^{s,t}(b) & \text{for } t \le d = s' \\ P_G^{s,d}(b) P_G^{d,t}(b) & \text{for } t > d = s' \end{cases} \tag{22}$$

Clearly, this is a special case of (21) with $b' = b$ and $\Delta_i = 1$. The chained index has a break at d. For instance, provided any transitive elementary index, we have, for $t > d$,

$$P^{s,t}(b) = \sum_i w_i^b P_i^{s,d} P_i^{d,t} = P^{s,d}(b) \left(\sum_i \tilde{w}_i^{b'} P_i^{d,t} \right) \neq P^{s,d}(b) P^{d,t}(b)$$

where $\tilde{w}_i^{b'} = \left(w_i^b P_i^{s,d}\right) / \left(\sum_k w_k^b P_k^{s,d}\right)$ are the price-updated weights from b to $b' = b + (d - s)$. It follows that, provided any transitive elementary index, one can avoid the break by using

$$\tilde{P}_G^t = \begin{cases} P_G^{s,t}(b) & \text{for } t \le d = s' \\ P_G^{s,d}(b) \tilde{P}_G^{d,t} & \text{for } t > d = s' \text{ and } \tilde{P}^{d,t} = \sum_i \tilde{w}_i^{b'} P_i^{d,t} \end{cases} \tag{23}$$

which is a special case of (21) with $b' = b$ and $\Delta_i = P_i^{s,d}$. Notice that the index head is given by $\tilde{P}^{d,t} = P^{s,t}(b)/P^{s,d}(b)$ which aims at $\theta^{s,t}(b)/\theta^{s,d}(b)$, wherever direct price comparisons between t and s are possible. Hence, I refer to (23) as *indirect* chaining, and the option is useful for updating *random* drop-outs of outlets and/or products that are unrelated to the price development. In contrast, we refer to Formula (22) as *direct* chaining, because the price development from d to t is measured *directly* by $P^{d,t}(b)$ which aims at $\theta^{d,t}(b)$.

Another aspect that is worth considering for the choice between the two chaining options is the "weights" for the index head, which is algebraically the difference between the two chained indices. For instance, if the weights are supposed to be good estimates of the values shares at the price reference period d, then one might choose between w_i^b and $\tilde{w}_i^{b+(d-s)}$ accordingly.

Finally, chaining of P-type indices can be given similarly: one only needs to replace every P with the corresponding $\wp$ in the formulae (22) and (23). Inaddition, for (23), we have

$$\tilde{\wp}^{d,t} = \left\{ \sum_i \tilde{w}_i^{b'} \left(P_i^{d,t} \right)^{-1} \right\}^{-1} \quad \text{where} \quad \tilde{w}_i^{b'} = \left(w_i^b / P_i^{s,d} \right) \Big/ \left(\sum_k w_k^b / P_k^{s,d} \right) \quad \text{and}$$

$$b' = b - (d - s)$$

4.3.3. Change in Weights Period

Consider updating the weights period from b to b', for $b < b'$. Break is then unavoidable. It is also highly common that a change in b comes together with a change of the price reference period from s to $s' = d$. For example, in a number of countries the chaining of the CPI takes place once a year with simultaneous updating of the weights and the reference prices. Put

$$\tilde{P}_G^t = \begin{cases} P_G^{s,t}(b) & \text{for } t \le d = s' \\ P_G^{s,d}(b) P_G^{d,t}(b') & \text{for } t > d = s' \end{cases} \tag{24}$$

which is a special case of (21) with $b' > b$ and $\Delta_i \equiv 1$. The target index parameter is given by $\theta^{s,d}(b)\theta^{d,t}(b')$, and we again refer to Formula (24) as *direct* chaining.

In theory, however, it may be possible to keep the price reference period fixed, as long as direct price comparisons from s to t are possible. Assume any transitive elementary index. Put

$$\tilde{P}_G^t = \begin{cases} P_G^{s,t}(b) & \text{for } t \le d \\ P_G^{s,d}(b) \tilde{P}_G^{d,t} & \text{for } t > d \end{cases} \quad \text{and} \quad \tilde{P}^{d,t} = \sum_i \tilde{w}_i^{b'+(d-s)} P_i^{d,t} \tag{25}$$

which a special case of (21) with $\tilde{w}_i^{b'+(d-s)} = \left(w_i^{b'} P_i^{s,d} \right) / \left(\sum_k w_i^{b'} P_k^{s,d} \right)$ and $\Delta_i = P_i^{s,d}$. We have $\tilde{P}_G^{d,t} = P^{s,t}(b')/P^{s,d}(b')$, such that the target index head parameter is given by $\theta^{s,t}(b')/\theta^{s,d}(b')$ instead of $\theta^{d,t}(b')$, and we may refer to Formula (25) as *indirect* chaining.

In practice the choice between (24) and (25) is often dictated by the necessity of updating the reference prices. In situations where both are feasible, the difference again comes down to the "weights" for the index head, i.e., $w_i^{b'}$ *vs.* $\tilde{w}_i^{b'+(d-s)}$, according to which a choice between the two needs to be made. Finally, chaining of P-indices can be given similarly: one only needs to replace every P with the corresponding $\wp$ in the

Formulae (24) and (25). In addition, for (25), we have

$$\tilde{\wp}^{d,t} = \left(\sum_i \tilde{w}_i^{b'-(d-s)} / P_i^{d,t} \right)^{-1} = \wp^{s,t}(b') / \wp^{s,d}(b')$$

$$\text{for } \tilde{w}_i^{b'-(d-s)} = \left(w_i^{b'} / P_i^{s,d} \right) / \left(\sum_k w_k^{b'} / P_k^{s,d} \right)$$

5. An Illustration

The variance estimation approach outlined above has been implemented in a generic SAS application PRIS at Statistics Norway, including the general formulae for fixed weights and chained indices. At the moment there are over 20 price statistics that use PRIS for production. For an illustration I present below some results on the quarterly Service Price Index for Industrial Cleaning (SPIIC), taken from Hayat and Sæter (2008).

The SPIIC covers industrial cleaning in NACE 74.7. There are about 2500 establishments that carry out the service in the population. A sample of about 90 establishments are selected by stratified sampling with disproportionate allocation of the stratum sample size. The larger establishments receive progressively higher inclusion probabilities, and the largest establishments are self-representing in the sample. In each period an establishment reports the prices of 3 self-selected contracts. Over time the contracts can be replaced by the respondents, on which occasion both the prices of the current and proceeding periods are required for the "new" contract.

The elementary groups are formed according to the size of the establishments. Each self-representing establishment forms an elementary aggregation on its own. The Jevons index is chosen to be the elementary index formula. The weights are obtained from the structural business survey and updated on a yearly basis. Higher level indices are the L-type indices.

Direct chaining is carried out in each period. Firstly, a *period* index is calculated from the proceeding period to the current period. This is the index head, which has shifting reference prices every time. Multiplication to the chained index from the preceding period then yields the chained index for the current period. In addition, once a year, the weights are updated, on which occasion the period index is calculated on the basis of the updated reference prices and weights, i.e., direct chaining. Under the approach of this paper, a variance estimate for the chained current index is then given by a weighted average of the variances in all the elementary period indices.

The chained index, the period index and the associated estimated standard error (SE) are all given in Table 1 for the SPIIC from the 1st quarter in 2005 to the 1st quarter in 2008. The estimated SEs are for the most part fairly stable throughout these quarters. A clear exception is the 2nd quarter in 2005 where the SE was over 5 times as large as the average SE for the whole series. Inspection of elementary indices and the associated SEs reveal considerable variation for the period index from the 1st to 2nd quarter in

Table 1. Chained index, period index and standard error for SPIIC

Quarter	Year	Chained Index	Period Index	Standard Error
1st	2005	0.966	1.0028	0.0023
2nd	2005	1.013	1.0493	0.0169
3rd	2005	1.014	1.0003	0.0030
4th	2005	1.007	0.9935	0.0026
1st	2006	1.009	1.0023	0.0035
2nd	2006	1.043	1.0328	0.0027
3rd	2006	1.052	1.0088	0.0027
4th	2006	1.062	1.0102	0.0031
1st	2007	1.077	1.0140	0.0015
2nd	2007	1.098	1.0188	0.0023
3rd	2007	1.111	1.0125	0.0015
4th	2007	1.113	1.0013	0.0009
1st	2008	1.142	1.0265	0.0019

2005: the elementary indices vary from about 0.325 to 5.505, and the associated SEs can be very large in cases where an elementary index deviates considerably from the unity. In retrospect it was clear that much of this was caused by inappropriate self-administered choices of contracts at this early age of the SPIIC. Not all the respondents had fully understood from the beginning the request to select contracts that are representative of price developments. This can been seen in the fact that there were many contract substitutions at the time. An unavoidable side-effect is a higher rate of missing price observations, which inflates the variance. In addition, labour cost is the most important component of price in industrial cleaning. Many establishments used to adjust the level of contract prices every year after the branch wage negotiation, which takes place during the 1st quarter of the year. A combination of these factors seemed to have caused a lot of "noise" (or divergence) in the observed price relatives for this period index. In the light of this, one may say that the model-based variance estimate did a good job in capturing this information, which was not possible in this case under a sampling-based approach due to the lack of a well-defined sampling design of contract prices.

6. A Final Remark

In the above I have presented a general model-based approach to variance estimation for fixed weights and chained price indices. The methodology has been implemented in a system application for statistical production at Statistics Norway. The approach covers most of the price index calculations in practice, providing a measure of uncertainty that is zero only if all the prices move in perfect unison or if, asymptotically, the number of observations tend to infinity in all the elementary groups. An important class of price indices that was not discussed here is the hedonic indices (see Triplett 2004). However, hedonic indices are after all distinctly stochastic in nature, such that the approach used here is applicable in rather a straight forward manner (see Zhang 2006). In this sense, variance estimation for the various types of price index calculations can now be brought under a unified perspective.

7. References

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