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The elasticity of substitution of superlative price indices

Abstract:
The paper presents a method for computing the curvature implicit in the use of superlative price indices. It extends the quadratic lemma and allows us to compute the elasticity of substitution of the underlying preferences in the direction of the observed price change for the Törnqvist and the quadratic mean of order r indices. It derives the expressions for the directional shadow elasticity of substitution and applies the results to the Norwegian CPI data base.

Keywords: elasticity of substitution, superlative index, consumer price index (CPI).

JEL classification: C43, D12

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1. Introduction

The theory of exact and superlative price indices has shown that we are able, in certain situations, to recover the change in the cost of living index from observations on prices and commodity purchases in the comparison and the base periods, data which are readily available to statistical agencies compiling cost of living indices. The applied work has used extensively on the family of quadratic mean of order r functions and in particular the special cases, the Törnqvist index and the Fisher ideal index, and the proof of their ability to recover the true index has relied on the quadratic (approximation) lemma which completely characterizes quadratic functions as shown by Diewert (1976, 2002).

The thrust of the present paper is that the quadratic lemma, together with the use of the Törnqvist or the quadratic mean of order r indices, is able to provide us with information about the second order or curvature properties of the underlying preferences. By using an extension of the quadratic lemma with the directional shadow elasticity of substitution (DSES), a concept introduced by the author in Frenger (1978, 1985), as the measure of curvature, the main result of the paper is the derivation of explicit formulae for the DSES of the underlying preferences for the translog and the quadratic mean of order r expenditure functions [see (23) and (33) below]. These elasticities, however, can only be computed in the direction of the observed price change, since that is all the data will permit. I have included some numerical examples of the method based on Norwegian data.

The paper is rather technical, but it would appear to have important implications for empirical work on price indices, as the numerical examples illustrate, particularly when one recognizes that the DSES may be interpreted as a local measure of the substitution bias. The last decade has seen a great interest in many countries in revising the methodology for computing the consumer price index (CPI) in a way that better allows for the commodity substitution which occurs as consumers alter their purchasing patterns in response to changing prices. The Boskin Commission (1996, p. iii) recommends that the "CPI should move toward a COLI concept by adopting a 'superlative' index formula to account for changing market baskets". More recently another committee, Schultze and Mackie (2002, p. 6), concludes that "The BLS should publish, contemporaneous with the real-time CPI, an advance estimate of the superlative index, utilizing either a constant-elasticity-of-substitution method or some other technique." Shapiro and Wilcox (1997) propose a method based on the CES function for extending superlative indices outside the observation period.

The discussion of superlative indices or flexible functional forms has often been presented as a choice between the Törnqvist (or translog) index and Fisher's ideal index. Alternatively one talks about substitution bias as the difference in measured inflation resulting from the use of the traditional Laspeyres index and either of the above indices. One of the problems with this debate is that we have no generally accepted way of measuring substitution or, as I tend to view it, the curvature of the underlying indifference surface along which the cost of living index (COLI) is measured. In a recent paper Frenger (2005) analyzes the definition of the substitution bias and the relationship between the bias and the elasticity of substitution.

The next three sections introduce the concepts needed to derive the main results of the paper. Section 2 reminds the reader of the definitions of exact and superlative indices and presents the quadratic mean of order r family of price indices. Section 3 introduces the quadratic lemma and an extension which permits us to characterize the second order properties of the quadratic function on the basis of the gradient only. In section 4 I define the directional shadow elasticity of substitution (DSES) which provides a measure of the elasticity of substitution of the underlying preferences for an arbitrary price change, such as the price change from the base period to the comparison period. With these tools at hand, we are then ready for the calculation of the expressions for the DSESs for the translog expenditure function (and the Törnqvist index) in section 5 and the quadratic mean of order r expenditure function in section 6. Each of these sections includes numerical examples based on Norwegian CPI data. The basic ideas and procedures are introduced in section 5 on the translog function and permit the derivation of the expression for the DSES in (23) and the calculation of the numerical example in the last column of table 1. The procedure is essentially repeated in the r-mean case (which in principle includes the translog function), but in this case we are also able to analyze the effect of altering the r parameter as illustrated in figure 2. The paper ends with some concluding comments.

2. Exact and superlative price indices

We will follow the standard practice in the applied theory of cost of living indices and assume there is a single representative consumer with homothetic preference, which may be represented dually by the expenditure function $C(u,p) = u\,c(p)$ giving the minimum expenditure necessary to reach the utility level u when the prices are $p = (p_1, \ldots, p_n)$. The function c is the unit expenditure function, which we assume to possess the standard neoclassical properties, in particular monotonicity, linear homogeneity, quasiconcavity, and twice continuous differentiability in p for $p > 0$. Let $c_p = [c_1, \ldots, c_n]$ and $c_{pp} = [c_{ij}]_{i,j=1,\ldots,n}$ denote the gradient and the Hessian matrix of the unit expenditure function. Shephard's lemma implies that the constant utility ($u = 1$) Hicksian demand function are $x_i = \partial c(p)/\partial p_i = c_i(p)$, $i = 1, \ldots, n$ or $x = c_p(p)$. Many of the expressions below will be simplified by the introduction of the value shares $s_i = p_i x_i / \sum_j p_j x_j$.

By definition the true (or Konüs) cost of living index,

$$P^*(p^1, p^0) = \frac{c(p^1)}{c(p^0)} ,$$

is the ratio of the minimum expenditure needed in the comparison period t_1 to maintain the same level of utility (in the present case $u = 1$) as in the base period t_0. The function c is in general unknown and we tend to approximate the true index with price indices of the form $P(p^1, x^1; p^0, x^0)$, which are functions of the price and consumption vectors in the two periods. These are typically the data which are available for the construction of consumer price indices from annual consumer surveys and monthly price sampling.

Occasionally we find ourself in the fortunate situation of being able to recover the true cost of living index:

Definition. Consider the unit expenditure function $c(p)$ and the price index function $P(p^1, x^1; p^0, x^0)$. The price index function P is *exact for c* if

$$P(p^1, x^1; p^0, x^0) \;=\; \frac{c(p^1)}{c(p^0)} \;,$$

for all p^1, p^2 in R^n_+ and $x^t = c_p(p^t)$, $t = 1, 2$.[1]

Note that it is required that the commodity demand vectors x^0 and x^1 be optimal at the respective prices.

The concept of an exact index is both intuitive and useful: the information provided by observations on price and quantity at two different periods is sufficient to recover the change in the value of the underlying function. As defined above the exact index is a function from R^{4n}_+ into R_+. In practice, however, the definition is also applied to functions defined on a subset of R^{4n}_+, and to functions defined on R^{4n}_+ and some additional parameter space.

An example of an exact index is provided by the CES index

$$P^{CES}(p^1, p^0, x^0) \;=\; \left[\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \;,$$

which is exact for the CES unit expenditure function with σ as substitution parameter. It was introduced by Lloyd (1975) and recently popularized by Shapiro and Wilcox (1997) as a way of constructing a real-time index which allows for substitution possibilities. Schultze and Mackie (2002) have recently recommended that the Bureau of Labor Statistics publish "an advance estimate of the superlative index, utilizing either a constant-elasticity-of-substitution method or some other technique." [2]

The introduction of the Generalized Leontief [Diewert (1971)] and the translog [Christensen, Jorgenson, and Lau (1971, 1973)] functional forms supplied the econometrician with functions that were richly enough parametrized to provide a second order approximation an arbitrary function at any given point. This led to the concept of flexibility of a function. A unit expenditure function c is *flexible* if its capable of providing a second order (differential) approximation to an arbitrary twice continuously differentiable unit expenditure function at any point in its domain. Extending this concept to indices we say that:

Definition. A price index $P(p^1, x^1; p^0, x^0)$ is *superlative* if it is exact for a flexible unit expenditure function c.

[1] See Diewert (1976). See also Diewert (1981), section 6 on "Superlative index number formulae" and Diewert's Palgrave article [Diewert (1987), pp. 772–3].

[2] Schultze and Mackie (2002), Executive Summary, p. 6, Recommendation 7-1.

Among the most widely used superlative indices are the The Törnqvist price index,

$$P^0(p^1, x^1; p^0, x^0) \;=\; \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0}\right)^{\frac{1}{2}(s_i^0+s_i^1)} , \qquad s_i^t = \frac{p_i^t x_i^t}{\sum_j p_j^t x_j^t} , \quad t = 0, 1, \tag{1}$$

and the quadratic mean of order r price index[3]

$$P^r(p^1, x^1; p^0, x^0) \;=\; \left[\frac{\displaystyle\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}}}{\displaystyle\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^0}{p_i^1}\right)^{\frac{r}{2}}}\right]^{\frac{1}{r}} , \qquad s_i^t = \frac{p_i^t x_i^t}{\sum_j p_j^t x_j^t} , \quad t = 0, 1, \quad r \neq 0. \tag{2}$$

These two indices will be analyzed at length in sections 5 and 6 below, where we also show that the indices themselves provide a second order approximation in the direction of the observed price change.

For $r = 2$, the quadratic mean of order 2 reduces to the Fisher ideal index

$$P^2(p^1, x^1; p^0, x^0) \;=\; \left[\frac{\displaystyle\sum_{i=1}^{n} p_i^1\, x_i^0}{\displaystyle\sum_{i=1}^{n} p_i^0\, x_i^0} \; \frac{\displaystyle\sum_{i=1}^{n} p_i^1\, x_i^1}{\displaystyle\sum_{i=1}^{n} p_i^0\, x_i^1}\right]^{\frac{1}{2}} ,$$

which is also the square root of the product of the Laspeyres and Paasche indices. The quadratic mean of order r price index P^r and the Törnqvist index P^0 are essentially members of the same family of superlative indices since P^0 is the limit of P^r as $r \to 0$.[4]

3. The quadratic lemma

The quadratic lemma was introduced into the theory of index numbers by Diewert (1976, p. 118), who calls it the "quadratic approximation lemma" even though no "approximation" seems to be involved. And in fact he uses it to prove that the Törnqvist index is exact. Once we know that the function is quadratic we know a good deal more about it than what is revealed by the quadratic lemma, and these facts are summarized in the extended quadratic lemma below.

Consider the quadratic function

$$f(z) \;=\; a_0 \;+\; a'z \;+\; \frac{1}{2}\, z'A\, z \;, \tag{3}$$

[3] See Diewert (1976, eq. 4.5, p. 131) and theorem 4.11, p. 133.

[4] Diewert (1980, p. 451) proves the result for the corresponding expenditure functions.

where a_0, $a = [a_i]$, and $A = [a_{ij}]$ are constants, and A is a symmetric matrix. The function f is defined for all $z \in R^n$. The gradient and the Hessian of the quadratic function (3) are given by:

$$\begin{aligned} f_z &= \nabla f = a + A\,z \ , \\ f_{zz} &= \nabla^2 f = A \ , \end{aligned}$$

the Hessian being in this case constant.

Assume that we are given two points z^0 and z^1 in R^n and the gradients $\nabla f(z^0)$ and $\nabla f(z^1)$ at these points. This information is essentially sufficient to characterize fully the behavior of the quadratic function f along the line

$$\begin{aligned} \ell &= \{\, z \mid z = z^0 + t\,(z^1 - z^0) \text{ for some } t \in R \,\} \ , \\ z(t) &= z^0 + t\,(z^1 - z^0) \ , \qquad\qquad t \in R \ , \end{aligned} \tag{4}$$

through z^0 and z^1, as formalized in lemma 2 below. To start with we may note that the expression for the gradient implies that

$$\nabla f(z^1) - \nabla f(z^0) = A\,(z^1 - z^0) \ , \tag{5}$$

showing that important information about the second order properties of the quadratic function f in the direction $z^1 - z^0$ is contained in the difference between the gradients at these points. In particular it implies that we need not know the elements of the A matrix.

The quadratic lemma demonstrates that we can recover the change in the value of the function f from the knowledge of the gradient at two different points in its domain, once we know that it is quadratic.[5]

Lemma 1. Quadratic lemma. Let f be a continuously differentiable function defined on $D \subset R^n$, and let $\nabla f(z^l)$ denote the gradient of f evaluated at $z^l \in D$, $l = 0, 1$. Then f is the quadratic function defined by (3) if and only if

$$f(z^1) - f(z^0) = \frac{1}{2}\left[\nabla f(z^0) + \nabla f(z^1)\right]' (z^1 - z^0) \ ,$$

for all z^0 and z^1 in D.[6]

Diewert (1976, p. 116), points out that the assumption that the function is quadratic and the use of the associated exact index implies that “we do not have to estimate the unknown coefficients in the A [coefficient] matrix.” We do in fact know exactly how much the function changes in value from z^0 to z^1. But we know more: we know the change in the value of the function along the whole line segment through z^0 and z^1, and we know the gradient and the “second derivative” along this line segment as the following lemma shows.

[5] The condition of the lemma was used extensively by Bowley (1928, p. 225), Frisch (1936, pp. 27–8), and Wald (1939, p. 321) in their attempts to develop a price index consistent with a non-homothetic utility function.

[6] For proof see Diewert (1976, p. 138) and Lau (1979).

Lemma 2. Extended quadratic lemma. Let f be the quadratic function (3), let z^0, $z^1 \in R^n$, and let $z(t) = z^0 + t\,(z^1 - z^0)$, $t \in R$, be a point on the line through z^0 and z^1. Then

1) the value of f at $z = z(t)$ is

$$f(z) - f(z^0) \;=\; t\,\nabla f(z^0)'(z^1 - z^0) \;+\; \frac{1}{2}\,t^2\big(\nabla f(z^1) - \nabla f(z^0)\big)'(z^1 - z^0)\;,$$

2) the gradient at f at z is

$$\nabla f(z) \;=\; \nabla f(z^0) \;+\; t\big(\nabla f(z^1) - \nabla f(z^0)\big)\;.$$

3) the covariant derivative of ∇f at z^0 in the direction $z^1 - z^0$ is

$$\big[\bar{\nabla}_{z^1 - z^0}\,\nabla f\big](z^0) \;=\; \nabla f(z^1) - \nabla f(z^0)\;.$$

The first two properties recover the change in the value of the function and the gradient of f along the whole line ℓ, not just at z^0 and z^1. The third property shows that we also know the "second derivative" of f along the line segment through z^0 and z^1.

To make more precise this notion of a directional second derivative we have to borrow some concepts from differential geometry. The first derivative of the real-valued function f at z is represented by the gradient $\nabla f(z) = (f_1(z), \ldots, f_n(z))$, which is a vector in R^n. Considered as a function of z, the gradient ∇f forms a vector field on R^n, which to each point z in R^n assigns the vector $\nabla f(z)$. Let us now consider an arbitrary vector field X on R^n, whose value at p is $X(z) = (X_1(z), \ldots, X_n(z))$, and ask how this vector field changes as we move away from z in some direction v. This is described by the covariant derivative of a vector field $X(z)$ in the direction v, where v is a tangent vector at z, i.e. a vector v attached to the point $z \in R^n$.

Definition. Let X be a vector field on R^n and let v be a tangent vector to R^n at the point z. Then the *covariant derivative* of X with respect to v is the tangent vector

$$\bar{\nabla}_v X \;=\; X(z + t\,v)'(0)$$

at the point z.[7]

In our case $X = \nabla f$ is the gradient of the quadratic function f, $X = a + Az$, and $v = z^1 - z^0$. And the covariant derivative of the gradient at z^0 with respect to v is

$$\begin{aligned}\bar{\nabla}_v X(z^0) \;&=\; X(z^0 + t\,(z^1 - z^0))'(0)\\ &=\; \frac{\partial}{\partial t}\Big(a \;+\; A\big(z^0 + t\,(z^1 - z^0)\big)\Big)\Big|_{t=0} \;=\; A\,(z^1 - z^0)\end{aligned}$$

[7] See O'Neill (1966, pp. 77–78) or Hicks (1965, p. 18). The notion of a covariant derivative belongs to the geometry of the space under consideration. Here we consider R^n with its standard Euclidean geometry.

$$= \nabla f(z^1) - \nabla f(z^0) \; . \tag{6}$$

The last step follows from (5), thus proving the third statement in lemma 2.

The result is primarily intended to show that the knowledge of $\nabla f(z^t)$, $t = 0, 1$, is sufficient to determine the second order properties along the line through z^0 and z^1. We will now proceed to show that in the case of superlative price indices the same information will be sufficient to compute the curvature of the price frontier in the direction of the observed price change. But first we have to determine how to measure this curvature.

4. The directional elasticity of substitution

The major deficiency of the Laspeyres index, considered as a cost of living index, is that it fails to take account of the fact that consumers adjust their purchases of goods and services to changing prices, buying more of what has become relatively cheaper and less of what has become relatively more expensive. This behavior is reflected implicitly in the curvature of the preference field, and explicitly in the price elasticities of demand for the various goods.

It would therefore seem to be of great importance to quantify these curvature properties, but in practice the discussion is largely limited to statements of the type "it is better to use the geometric average rather than the arithmetic average because the elasticity of substitution is probably closer to unity than it is to zero".

There are at least two reasons for this state of affairs: (i) there is no generally accepted definition of the elasticity of substitution, and (ii) the elasticity of substitution, being a second order parameter, is difficult to measure. In the following we will overcome these difficulties by (i) choosing the directional shadow elasticity of substitution as the definition of curvature, and (2) demonstrating that the magnitude of the curvature follows directly from the choice of price index and the data (p^0, x^0) and (p^1, x^1) used to compute the index.

But let us first consider two traditional definitions of the elasticity of substitution. Since we are primarily dealing with price indices, it seems reasonable to restrict the presentation to measures of substitution in the price space. Let $p \in R^n_+$ be a price vector and u the utility level of a consumer.

1) *The partial (Allen-Uzawa) elasticity of substitution* of the i'th and the j'th commodities,

$$\mathrm{AUES}_{ij}(p) \;=\; \frac{c_{ij}(p)\, c(p)}{c_i(p)\, c_j(p)} \; , \qquad\qquad i, j = 1, \ldots, n \, . \tag{7}$$

   The AUES is essentially a renormalization of the elasticity of the Hicksian (or income compensated) demand for the i'th good with respect to the j'th price. The $[\mathrm{AUES}_{ij}]$ matrix has the advantage of being symmetric.

2) The *shadow elasticity of substitution* (SES) between commodity i and commodity j at p was defined by McFadden (1963) as the negative of the elasticity of the

commodity ratio $x_i/x_j = c_i(p)/c_j(p)$ with respect to a change in the price ratio p_i/p_j holding all other prices, and total expenditure constant ($i \neq j$),

$$\sigma_{ij}(p) \equiv - \left. \frac{\partial \ln \dfrac{x_i}{x_j}}{\partial \ln \dfrac{p_i}{p_j}} \right|_{\substack{c \text{ and } p_k, \\ k \neq i,j, \text{ const.}}} = \frac{- \dfrac{c_{ii}(p)}{c_i^2(p)} + 2 \dfrac{c_{ij}(p)}{c_i(p)\, c_j(p)} - \dfrac{c_{jj}(p)}{c_j^2(p)}}{\dfrac{1}{p_i\, c_i(p)} + \dfrac{1}{p_j\, c_j(p)}} . \tag{8}$$

Before introducing the third definition we need to define the *price frontier*

$$M = \{ p \mid c(p) \text{ is constant} \} , \tag{9}$$

as a level surface of the unit expenditure function. There is a separate price frontier for each value of the constant and there is a price frontier through each p. For a given $p \in M$ there is a set of price changes v which will leave total expenditure unchanged. This defines the *tangent plane to* M *at* p,

$$T_pM = \{ v \mid x'\, v = 0,\ x = c_p(p) \} . \tag{10}$$

If we identify v with a price change $\hat{p} - p$, then T_pM becomes the set of prices $\hat{p}$ which leave the Laspeyres index based at p unchanged.

Definition. Let c be a unit expenditure function and M the price frontier of c through p. Further, let v be a direction vector in the tangent plane T_pM to M at p. The *directional shadow elasticity of substitution* (DSES) at p in the direction v is [8]

$$\text{DSES}_p(v) = - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(p)\, v_i\, v_j}{\sum_{i=1}^{n} c_i(p)\, v_i\, \dfrac{v_i}{p_i}} , \qquad \begin{array}{l} p \in R_+^n , \\ v \in T_pM,\ v \neq 0 . \end{array} \tag{11}$$

Any price change $v \in T_pM$ will by definition leave total expenditures unchanged. The DSES is taken to be our measure of the curvature of the price frontier at p in the direction $v \in T_pM$.

Returning to the Allen-Uzawa elasticity we note that it implies a change in the j'th price only, and the associated price change vector v does not lie in T_pM.[9] The shadow elasticity of substitution σ_{ij}, on the other hand, is obtained as a special case of $\text{DSES}_p(v)$

[8] The DSES was introduced in Frenger (1978), which also presents the empirical application which motivated the definition. Frenger (1985) introduced the definition (11) and uses the DSES to test for the concavity of the underlying cost function. A more detailed presentation is given in Frenger (1992). There is a dual directional direct elasticity of substitution defined in the quantity space.

[9] In the biased opinion of the author, the partial elasticity of substitution (7) is not a proper measure of curvature, and therefore not a proper elasticity of substitution. The same malaise affects also the Morishima elasticities!

by choosing v such that $c_i v_i + c_j v_j = 0$, and $v_k = 0$ for $k \neq i, j$: only the i'th and the j'th prices change and total expenditure is constant. This price change v thus lies in $T_p M$.

The major advantage of the DSES is that it is defined for an arbitrary price change. In the context of price indices and homothetic preference it will allow us to measure the elasticity of substitution in the direction of the actual price change from one period to the other, f.ex. from the base period 0 to the comparison period 1. When there are only two commodities or two prices ($n = 2$), all three definitions of the elasticity of substitution coincide (as do most other definitions), and there is but a single measure of the elasticity of substitution.

The current application of the DSES is somewhat similar to the situation which led to the introduction of the DSES in Frenger (1978) since we are given an explicit (historical) price change and want to measure the curvature in the direction of that price change. In the 1978 paper I was estimating a two level production structure and wanted to measure the consequence which the use of inconsistent aggregates at the lower level had upon the upper level measure of substitution. And since the magnitude of the effect depended on the direction of change, it seemed most appropriate to measure the effect along the actual price change.

5. The translog function

The translog function was introduced by Christensen, Jorgenson, and Lau (1971, 1973), while the Törnqvist price index was introduced by Törnqvist back in 1936 as a discrete approximation to a Divisia index.[10] Diewert (1976) showed that the Törnqvist price index is exact for the homogeneous translog expenditure function.

5.1. The translog unit expenditure function

Consider the log of the translog unit expenditure function

$$\ln c(p) \;=\; \alpha_0 \;+\; \sum_{i=1}^{n} \alpha_i \ln p_i \;+\; \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln p_i \ln p_j \;, \tag{12}$$

with $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{j=1}^{n} \gamma_{ij} = 0$, $i = 1, \ldots, n$, and $\gamma_{ij} = \gamma_{ji}$.

The first derivatives of the translog unit expenditure function are

$$c_i(p) \;=\; \frac{\partial\, c(p)}{\partial\, p_i} \;=\; \frac{c}{p_i} \left(\alpha_i + \sum_{j=1}^{n} \gamma_{ij} \ln p_j \right) \;, \qquad i = 1, \ldots, n.$$

[10] Törnqvist (1936) introduced the weighted geometric average, chaining the index and allowing weights which are "variable in principle". The weighting scheme $(s_i^0 + s_i^1)/2$ does not appear in the article.

The value shares become

$$s_i \;=\; \frac{p_i c_i}{c} \;=\; \alpha_i \;+\; \sum_{j=1}^{n} \gamma_{ij} \ln p_j \;, \qquad\qquad i = 1, \ldots, n.$$

The second derivatives are given by

$$c_{ij}(p) \;=\; \frac{\partial^2 c(p)}{\partial p_i\, \partial p_j} \;=\; \gamma_{ij}\, \frac{c}{p_i\, p_j} \;+\; \frac{c_i\, c_j}{c} \;-\; \frac{c_i}{p_i}\, \delta_{ij} \;, \qquad\qquad i,j = 1, \ldots, n, \tag{13}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

We obtain the directional shadow elasticity of substitution of the translog function (12) at p in the direction $v \in T_pM$ by substituting for the derivatives c_i and c_{ij} in (11),

$$\mathrm{DSES}_p(v) \;=\; -\,\frac{\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, v_i\, v_j}{\displaystyle\sum_{i=1}^{n} c_i\, v_i\, \frac{v_i}{p_i}} \;=\; 1 \;-\; \frac{\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}\, \frac{v_i}{p_i}\, \frac{v_j}{p_j}}{\displaystyle\sum_{i=1}^{n} s_i\, \frac{v_i^2}{p_i^2}} \;, \qquad \begin{array}{l} p \in R_+^n, \\ v \in T_pM,\;\; v \neq 0\,. \end{array} \tag{14}$$

The expression still depends on all the second order parameters γ_{ij} of the translog function. The translog function reduces to the Cobb-Douglas function when $\gamma_{ij} = 0$ for all i and j. And we see from (14) that in this case the DSES is unity in all directions, as expected.

5.2. The logarithmic transformation

Assume that we are given the prices p^0 and p^1 and the quantity demanded x^0 and x^1 at two different periods in time, $t = 0, 1$. Since we are using the unit expenditure function, we are assuming that the level of utility is constant and equal to one. Only prices change, and with them the optimal commodity vector.

The quadratic approximation lemma 1 and lemma 2 show that we can deduce a great deal about the behavior of the function from information on its gradient at two points in its domain, if we know that the function is quadratic. We will now apply many of the same ideas to the translog function by utilizing the fact that a simple transformation of the coordinate axes will change the translog into a quadratic function.

Applying the transformations[11]

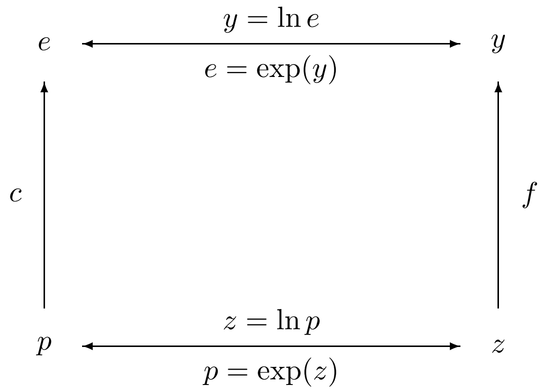
$$y \;=\; \ln e\,, \qquad\qquad \text{and} \qquad\qquad z_i \;=\; \ln p_i\,, \qquad\qquad i = 1, \ldots, n, \tag{15}$$

from (e, p) space into (y, z) space to the translog function converts it into the quadratic function [see (3)]

$$y \;=\; f(z) \;=\; \alpha_0 \;+\; \sum_{i=1}^{n} \alpha_i\, z_i \;+\; \frac{1}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}\, z_i z_j \;.$$

The relationship between the (e, p) and the (y, z) spaces is illustrated in figure 1.

Figure 1: The translog transformation



We can express the relationship between y and z directly in terms of the quadratic f function, or indirectly via the unit expenditure function. And equivalently for the relationship between e and p:

$$\begin{aligned} y &= f(z) = \ln c(\exp[z]) \ , \\ e &= c(p) = \exp[f(\ln p)] \ . \end{aligned}$$

Differentiating the first identity gives the relationship between the gradients of f and c,

$$\nabla f(z) \; = \; \frac{1}{c(p)} \, \hat{p} \, \nabla c(p) \; = \; s(p) \ ,$$

where $\hat{p}$ denotes a diagonal matrix with p on the diagonal and s is the vector of value shares.

Application of the quadratic lemma 1 then yields

$$\ln c(p^1) \; - \; \ln c(p^0) \; = \; \frac{1}{2} \left[s^0 + s^1 \right]' (\ln p^1 - \ln p^0) \ ,$$

or

$$\frac{c(p^1)}{c(p^0)} \; = \; \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0} \right)^{\frac{1}{2}(s_i^0 + s_i^1)} \; = \; P^0(p^1, x^1; p^0, x^0) \ .$$

The expression on the right is the Törnqvist price index (1), and the derivation shows that it is exact for the translog unit expenditure function.[12]

Further (5) implies that ($\Gamma = [\gamma_{ij}]$)

$$s^1 - s^0 \; = \; \Gamma \, (\ln p^1 - \ln p^0) \ , \tag{16}$$

which may be verified by direct computation on the translog function.

[11] See Diewert (1976, p. 119).

[12] See Diewert (1976, eq. 2.15, p. 121).

5.3. The price curves

Applying lemma 2 allows us to compute the value of the translog unit expenditure function along the curve $\beta(t)$,

$$p(t) \;=\; \beta(t) \;=\; \exp\bigl[\ln p^0 + t\,(\ln p^1 - \ln p^0)\bigr] \;=\; p^0\, e^{t\,(\ln p^1 - \ln p^0)} \;, \tag{17}$$

with components

$$p_i(t) \;=\; p_i^0 \left(\frac{p_i^1}{p_i^0}\right)^t , \qquad\qquad i = 1, \ldots, n.$$

Note that $\beta(0) = p^0$ and $\beta(1) = p^1$. Each component of the price vector along β grows at a constant rate $\ln(p_i^1/p_i^0)$. The tangent to this curve at p^0 is

$$\bar{v} \;=\; \beta'(0) \;=\; \left(p_1^0 \ln\frac{p_1^1}{p_1^0}, \ldots, p_i^0 \ln\frac{p_i^1}{p_i^0}, \ldots, p_n^0 \ln\frac{p_n^1}{p_n^0}\right) \;=\; \hat{p}^0\,(\ln p^1 - \ln p^0) \;. \tag{18}$$

Why do we use the direction vector $\bar{v}$ instead of computing the simpler, and perhaps more intuitive, direction vector $\tilde{\bar{v}} = p^1 - p^0$? The answer is provided by the quadratic lemma: $\bar{v}$ is the only direction in which we can compute the change in the value of the function and the DSES on the bases of the information at hand. A justification for the procedure is provided by the observation that $\bar{v}$ will be close to $\tilde{\bar{v}}$ if the price change is small. A second argument is that we observe the prices p^0 and p^1 at two different points in time, but we have no information about how the change from p^0 to p^1 occurred while what we really are interested in is the *initial* direction of change of prices as we move from t_0 to t_1. This initial direction of change is probably neither $\bar{v}$ nor $\tilde{\bar{v}}$, and may not be economically well-defined at all!

The value of the unit expenditure function will not be constant along the curve $\beta(t)$, in fact it will not be so even if $c(p^0) = c(p^1)$ unless we impose severe restrictions on the coefficient matrix $\Gamma = [\gamma_{ij}]$. We can, however, construct a new curve α along which costs are constant, and thus lies in the $c(p^0)$ price frontier, by using the homogeneity of the expenditure function to proportionately adjust all prices, and define the curve

$$\alpha(t) \;= (\tilde{p}_1, \ldots, \tilde{p}_i, \ldots, \tilde{p}_n)(t) \;=\; \frac{c(p^0)}{c(\beta(t))}\,\beta(t) \;, \tag{19}$$

the coordinates of the curve being

$$\tilde{p}_i(t) \;=\; \frac{c(p^0)}{c(p(t))}\, p_i^0 \left(\frac{p_i^1}{p_i^0}\right)^t , \qquad\qquad i = 1, \ldots, n\,.$$

The curve will not pass through p^1 unless $c(p^1) = c(p^0)$.

The coordinates of the velocity vector along α are

$$v_i(t) \;=\; \frac{d\,\tilde{p}_i(t)}{d\,t} \;=\; \tilde{p}_i(t)\left[\ln\frac{p_i^1}{p_i^0} \;-\; \sum_{k=1}^{n} s_k(t)\,\ln\frac{p_k^1}{p_k^0}\right] , \tag{20}$$

while the components of the initial velocity at p^0 are

$$v_i \;=\; \left.\frac{d\,\tilde{p}_i(t)}{d\,t}\right|_{t=0} \;=\; p_i^0\left[\ln\frac{p_i^1}{p_i^0} \;-\; \sum_{k=1}^{n} s_k^0\,\ln\frac{p_k^1}{p_k^0}\right]\,, \qquad\qquad i=1,\ldots,n.$$

The initial velocity of the curve $\alpha(t)$ at p^0 is then

$$v=(v_1,\ldots,v_i,\ldots,v_n) \;=\; \bar{v} \;-\; \rho^0\,p\;, \tag{21}$$

where we have introduced the *local deflation factor*

$$\rho^0 \;=\; \sum_{i=1}^{n} s_i^0\,\ln\frac{p_i^1}{p_i^0}\;. \tag{22}$$

It is readily verified that $v'x=0$, i.e. that v lies in tangent plane $T_{p^0}M$ to M at p^0. The factor ρ^0 represents the proportionate change in the prices at p^0 which would leave unit expenditure unchanged.

5.4. The DSES along the curve

The DSES for the translog function for an arbitrary price change $v \in T_pM$ was computed in (14), an expression which depends however on all the γ_{ij} parameters of the translog function. We will now evaluate the DSES at p^0 in the direction v determined by the price change from p^0 to p^1 and given by (21). Let us define

$$\pi_i \;=\; \ln\frac{p_i^1}{p_i^0}\;, \qquad\qquad i=1,\ldots,n,$$

and write

$$\frac{v_i}{p_i^0} \;=\; \pi_i \;-\; \rho^0\;.$$

Substituting for v_i/p_i^0 in the numerator of the expression for the DSES yields

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\,\frac{v_i}{p_i^0}\,\frac{v_j}{p_j^0} \;&=\; \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\,(\pi_i-\rho^0)\,(\pi_j-\rho^0)\\
&=\; \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\,\pi_i\pi_j \;-\; \rho^0\sum_{i=1}^{n}\pi_i\sum_{j=1}^{n}\gamma_{ij} \;-\; \rho^0\sum_{j=1}^{n}\pi_j\sum_{i=1}^{n}\gamma_{ij} \;+\; (\rho^0)^2\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\,\pi_i\pi_j \;=\; (\ln p^1-\ln p^0)'\,\Gamma\,(\ln p^1-\ln p^0)\\
&=(s^1-s^0)'\,(\ln p^1-\ln p^0)\;.
\end{aligned}$$

The last three terms in the second line vanish because $\sum_{i=1}^{n}\gamma_{ij}=\sum_{j=1}^{n}\gamma_{ij}=0$. In the last step I have utilized (16) which allows us to get rid of the elements of the unknown

$\Gamma = [\gamma_{ij}]$ matrix. This is the key step where we are able to replace the unknown second order parameters by the observed first order variables.

The denominator evaluated at p^0 in the direction (21) is

$$\sum_{i=1}^{n} s_i^0 \frac{v_i^2}{(p_i^0)^2} = \sum_{i=1}^{n} s_i^0 (\pi_i^2 - 2\rho^0 \pi_i + (\rho^0)^2) = \sum_{i=1}^{n} s_i^0 \pi_i^2 - (\rho^0)^2$$
$$= \sum_{i=1}^{n} s_i^0 \left(\ln \frac{p_i^1}{p_i^0}\right)^2 - \left(\sum_{k=1}^{n} s_k^0 \ln \frac{p_k^1}{p_k^0}\right)^2 .$$

Combining the two expressions gives the DSES of the translog function at p^0 in the direction $v \in T_{p^0}M$,

$$\mathrm{DSES}_{p^0}(v) = 1 - \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \frac{v_i}{p_i^0} \frac{v_j}{p_j^0}}{\sum_{i=1}^{n} s_i^0 \left(\frac{v_i}{p_i^0}\right)^2} = 1 - \frac{\sum_{i=1}^{n} (s_i^1 - s_i^0)(\ln p_i^1 - \ln p_i^0)}{\sum_{i=1}^{n} s_i^0 \left(\ln \frac{p_i^1}{p_i^0} - \sum_{k=1}^{n} s_k^0 \ln \frac{p_k^1}{p_k^0}\right)^2} . \tag{23}$$

We have succeeded in expressing the directional shadow elasticity of substitution at p^0 in the transformed direction of the observed price change $\bar{v} = \hat{p}^0 (\ln p^1 - \ln p^0)$ [see (18)] entirely in terms of the observed prices and quantities in the two periods $t = 0, 1$. This was only possible because the observed change in the gradient between p^0 and p^1 contains sufficient information about the curvature of the preferences (and the factor price frontier) in that direction, and only in that direction, and given the fact that we assumed the expenditure function to be homogeneous translog.[13]

5.5. Numerical illustration: The Norwegian CPI

Let us now illustrate the computation of the directional shadow elasticity of substitution implicit in the use of a Törnqvist price index. The data are provided by the database for the Norwegian CPI at the "3-digit" level. This is the lowest level for which the Consumer Survey provides the necessary budget shares. At this level the prices are elementary price indices computed from a selection of representative commodities. At the 3-digit level there were 148 commodities, but only 140 of these contained complete prices and weights information for the full period 1990–1998. Thus our price space has dimension 140.

R_+^{140} is a rather large space and one in which we are not used to talk about elasticities of substitution. And yet the argument above shows that with the implicit assumption of homogeneous translog preferences, the observations on prices and quantities (p^t, x^t) and (p^τ, x^τ) for any two years $t, \tau \in (1990, \ldots, 1998)$, $t \neq \tau$, allows us to compute the

[13]Returning to the Cobb-Douglas function, we know that in this case the value shares are constant, i.e. $s^0 = s^1$. The numerator in (23) is zero and $\mathrm{DSES}_{p^0}(v) = 1$.

directional shadow elasticity at p^t in the direction p^τ, or alternatively the DSES at p^τ in the direction p^t. These two elasticities would be computed at two different points on the same implicit translog function. The selection of two other observations $t', \tau' \in (1990, \ldots, 1998)$ would define a different translog function.

In table 1 we present four different price indices (Laspeyres, chained Laspeyres, Törnqvist, and Paasche) for the period 1990 to 1998. The indices are all normalized to unity in 1990. The fifth column presents the DSES associated with the Törnqvist index, each elasticity being computed at p^t in the direction p^{t+1}. The relationship between the indices

Table 1: Select price indices and the DSES

year	Laspeyres	Chained Laspeyres	Törnqvist	Paasche	DSES
1990	1.0000	1.0000	1.0000	1.0000	0.6760
1991	1.0336	1.0336	1.0328	1.0319	0.3467
1992	1.0579	1.0575	1.0565	1.0553	0.6702
1993	1.0839	1.0812	1.0798	1.0788	0.9794
1994	1.1014	1.0963	1.0941	1.0934	0.7941
1995	1.1303	1.1228	1.1201	1.1175	1.2739
1996	1.1419	1.1366	1.1332	1.1288	1.0808
1997	1.1726	1.1656	1.1616	1.1532	1.2552
1998	1.2014	1.1922	1.1867	1.1732	

is the expected one. We see from the table that the Paasche index is always lower than the Laspeyres index as "required" by the Paasche-Laspeyres bounding theorem, though strictly speaking it only applies to binary comparisons and it requires that the preferences be concave. The chained Laspeyres, which is essentially the index used in the Norwegian CPI, and the Törnqvist indices lie between the other two. The chained Laspeyres and the Törnqvist indices also provide us with an estimate of the substitution bias which amounts, on the average, to only 0.06% over the period.[14] As mentioned in the introduction, the DSES may be considered as a local measure of the substitution bias, while the substitution bias itself depends on both the curvature (or the DSES) and the size of the price change.

Returning to the column for the DSES we see that the directional shadow elasticity of substitution at $t_0 = 1990$ in the direction of the price change which occurred between 1990 and 1991 is 0.6760. It is computed on the basis of the translog expenditure function determined by the 1990 and 1991 observations obtained from the CPI data base. Similarly 0.3467 is the DSES for 1991 in the direction of the 1991–1992 price change as determined by the translog function based on the 1991 and 1992 observations. The DSES appearing in the table are thus computed at different points in the price space and for different

[14]The same low value is obtained in Frenger (2005), where the result is attributed to the annual rebasing of the Norwegian CPI and the modest annual change in relative prices.

translog functions. The DSESs are all positive, indicating that the underlying preferences are concave at the observation points and in the directions of the observed price changes.

As suggested above, we can pick any two years t and τ, determine the implied translog function, and then compute the DSES at either t or τ in the direction of the price change between the two years. This is done in table 2. The entry in the first row, second column, shows that the DSES "from 1990 to 1991", i.e. the DSES for 1990 in the direction of the 1990–1991 price change is 0.6760, as we know from the previous table. Similarly the entry on the first row, third column, shows that the DSES "from 1990 to 1992" is 0.4500. The DSES column from table 1 reappears on the "subdiagonal" above the empty diagonal.

Table 2: The DSES between t and τ, the translog function[15]

	comparison year								
base year	1990	1991	1992	1993	1994	1995	1996	1997	1998
1990	–	0.6760	0.4500	0.5287	0.6083	0.6711	0.5710	0.6320	0.6977
1991	0.6818	–	0.3467	0.5158	0.6485	0.6892	0.5826	0.6132	0.6972
1992	0.4577	0.3500	–	0.6702	0.7208	0.7255	0.6453	0.6309	0.7171
1993	0.5081	0.4928	0.6572	–	0.9794	0.8610	0.7957	0.7725	0.8910
1994	0.5659	0.6113	0.6914	0.9782	–	0.7941	0.9126	0.9803	1.1425
1995	0.6279	0.6477	0.6904	0.8490	0.7908	–	1.2739	1.3580	1.4501
1996	0.5345	0.5465	0.6158	0.7895	0.9140	1.2672	–	1.0808	1.2846
1997	0.6177	0.5950	0.6150	0.7740	0.9812	1.3394	1.0781	–	1.2552
1998	0.6894	0.6873	0.7099	0.8938	1.1323	1.4157	1.2725	1.2551	–

Any two years t and τ determine a unique translog function (restricted to the curve joining the observations for the two years). On this function we can determine the DSES at t in the direction τ, or we can determine the DSES at τ in the direction t. These two DSESs will in general not be equal since they are measured at two different points on the same function. In the table the DSESs for $t < \tau$ are shown above the diagonal, while those for $t > \tau$ are shown below it. Thus f.ex. column one, row two, shows that the DSES from 1991 in the direction 1990 is 0.6818, which is rather close to the value of 0.6760 we obtained for the DSES from 1990 to 1991.

Considering the extreme dates we observe that the DSES implicit in the price change from 1990 to 1998 is 0.6977 while going in the opposite direction from 1998 to 1990 gives a DSES of 0.6894. On the whole we see that these pairs of DSESs are all rather close, deviating from each other by at most 7%. This suggests that the variability of the DSES as we observe it in table 2 is mainly due to the fact that it is based on different translog functions, rather than being measured at different points on the same function.

[15] The elements on the diagonal are missing since the DSES is not defined.

It should also be noted that the DSESs are all positive as required by the theory. This shows that the underlying translog function is concave in the only direction in which it can be determined, and suggests that the data do result from an actual expenditure minimization problem or from a set of expenditure minimization problems. In a sense, the results are also an indication that the quality of the data is good and well suited for the computation of a CPI.

6. The quadratic mean of order r function

This section considers the quadratic mean of order r unit expenditure functions and the associated quadratic mean of order r price index which is exact for this functional form. We then derive the directional elasticity of substitution implicit in the use of this index. The procedure follows closely that of the translog function of the previous section, allowing us to leave out some steps. Compared with the translog function, there is one significant difference. The r mean function has one additional parameter, the exponent r itself, which must be specified *a priori*. The translog case is a special case of the r mean function, being the limit as $r \to 0$. This makes it possible to consider how the implicit DSES depend on the r parameter. At the end of the section we continue the numerical example based on the Norwegian CPI.

6.1. The quadratic mean of order r unit expenditure function

The quadratic mean of order r unit expenditure function is[16]

$$c_r(p) \;=\; \left[\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\, p_i^{r/2} p_j^{r/2}\right]^{1/r} , \qquad\qquad b_{ij} = b_{ji}\,,\; r \neq 0\,. \tag{24}$$

The subscript r denoting the order of the mean will generally be ignored in the following in order to simplify the notation. The subscripts i and j will be used to designate derivation with respect to the price arguments.

The first derivative of the unit expenditure function $c_r(p) = c(p)$ with respect to p_i is

$$c_i(p) \;=\; \frac{\partial\, c(p)}{\partial\, p_i} \;=\; c^{1-r} \sum_{j=1}^{n} b_{ij}\, p_i^{\frac{r}{2}-1} p_j^{\frac{r}{2}} \;, \qquad\qquad i = 1, \ldots, n,$$

while the value shares are given by

$$s_i(p) \;=\; \frac{p_i c_i}{c} \;=\; c^{-r} \sum_{j=1}^{n} b_{ij}\, p_i^{\frac{r}{2}} p_j^{\frac{r}{2}} \;, \qquad\qquad i = 1, \ldots, n.$$

[16]See Denny (1974, p. 26) or Diewert (1976, p. 130). It is common to designate the function by c_r to make explicit the order of the quadratic mean involved. In the derivation below we have however already sufficient subscripts and superscripts!

The second derivatives are

$$c_{ij}(p) \;=\; (1-r)\,\frac{c_i c_j}{c} \;+\; \frac{r}{2}\,c^{1-r}\,b_{ij}\,p_i^{\frac{r}{2}-1}p_j^{\frac{r}{2}-1} \;+\; \left(\frac{r}{2}-1\right)\frac{c_i}{p_i}\,\delta_{ij}\;. \tag{25}$$

Substituting these derivatives into (11) gives the directional shadow elasticity of substitution of the quadratic mean of order r function at p in the direction $v \in T_pM$,

$$\mathrm{DSES}_p(v) \;=\; \left(1-\frac{r}{2}\right) \;-\; \frac{r}{2}\,\frac{\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\left(\frac{p_i}{c}\,\frac{p_j}{c}\right)^{\frac{r}{2}}\frac{v_i}{p_i}\,\frac{v_j}{p_j}}{\displaystyle\sum_{i=1}^{n} s_i\,\frac{v_i^2}{p_i^2}}\;, \qquad \begin{array}{l} p \in R_+^n, \\ v \in T_pM,\; v \neq 0\,, \end{array} \tag{26}$$

This expression still depends on all the b_{ij} parameters of the quadratic mean function.

In part as a check on the computations, consider the CES case for which $b_{ij} = 0$ for $i \neq j$. Then

$$c_i \;=\; b_{ii}\left(\frac{p_i}{c}\right)^{r-1}, \qquad\qquad s_i \;=\; \frac{p_i c_i}{c} \;=\; b_{ii}\left(\frac{p_i}{c}\right)^{r},$$

and the directional shadow elasticity of substitution reduces to

$$\mathrm{DSES}_p(v) \;=\; \left(1-\frac{r}{2}\right) \;-\; \frac{r}{2}\,\frac{\displaystyle\sum_{i=1}^{n} b_{ii}\left(\frac{p_i}{c}\right)^{r}\frac{v_i^2}{p_i^2}}{\displaystyle\sum_{i=1}^{n} s_i\,\frac{v_i^2}{p_i^2}} \;=\; 1-r\;.$$

The DSES is constant for all $v \in T_pM$, as it should be for a constant elasticity of substitution function.

6.2. The exponential transformation

Assume that we are given the prices p^0 and p^1 and the quantity demanded x^0 and x^1 at two different periods in time, $t = 0, 1$. Since we are using the unit expenditure function, we are assuming that the level of utility is constant and equal to one. Only prices change, and with them the expenditure level.

The quadratic lemma 1 and lemma 2 showed that we could deduce a great deal about the behavior of the function from information on the gradient of the function at two points in its domain, if we knew that the function was quadratic. We will now apply many of the same ideas to the quadratic mean of order r function by utilizing the fact that a simple transformation of the coordinate axes will change the r mean into a quadratic function.

Applying the transformations

$$y \;=\; e^r\,, \qquad\qquad \text{and} \qquad\qquad z_i \;=\; p_i^{r/2}\,, \qquad\qquad i = 1, \ldots, n, \tag{27}$$

gives the quadratic function [see (3)]

$$y \;=\; f(z) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\, z_i z_j \;,$$

to which we can apply the quadratic lemma 1. Compared with the translog function, we may note the absence of a constant and the first order parameters. On the other hand there are no row sum restrictions on the b_{ij} parameters.

We can express the relationship between y and z directly in terms of the quadratic f function, or indirectly via the unit expenditure function:

$$y \;=\; f(z) \;=\; \left[c(z^{2/r})\right]^r \;.$$

Differentiating the first identity gives the relationship between the gradients of f and c,

$$\nabla f(z) \;=\; 2\, c^{r-1}\, \hat{p}^{1-\frac{r}{2}}\, \nabla c \;=\; 2\, c^r\, \hat{p}^{-\frac{r}{2}}\, s \;, \tag{28}$$

where $\hat{p}$ denotes a diagonal matrix with p on the diagonal and s is the vector of value shares.

Applying the quadratic lemma 1 then yields

$$c(p^1)^r \;-\; c(p^0)^r \;=\; \left[c(p^1)^r\,(\hat{p}^1)^{-\frac{r}{2}}\, s^1 \;+\; c(p^0)^r\,(\hat{p}^0)^{-\frac{r}{2}}\, s^0\right]' \left[(p^1)^{\frac{r}{2}} - (p^0)^{\frac{r}{2}}\right] \;,$$

which reduces to

$$c(p^1)^r\, s^{1\prime}\,(\hat{p}^1)^{-\frac{r}{2}}\,(p^0)^{\frac{r}{2}} \;=\; c(p^0)^r\, s^{0\prime}\,(\hat{p}^0)^{-\frac{r}{2}}\,(p^1)^{\frac{r}{2}} \;,$$

or [see (2)]

$$\frac{c(p^1)}{c(p^0)} \;=\; \left[\frac{\displaystyle\sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}}}{\displaystyle\sum_{i=1}^{n} s_i^1 \left(\frac{p_i^0}{p_i^1}\right)^{\frac{r}{2}}}\right]^{\frac{1}{r}} \;=\; P^r(p^1, x^1; p^0, x^0) \;, \tag{29}$$

which is the quadratic mean of order r price index (2). The derivation shows that it is exact for the quadratic mean of order r unit expenditure function.[17]

It will be convenient to define the deflation terms

$$\tilde{\rho}^0 \;=\; \sum_{k=1}^{n} s_k^0 \left(\frac{p_k^1}{p_k^0}\right)^{\frac{r}{2}} \qquad \text{and} \qquad \tilde{\rho}^1 \;=\; \sum_{k=1}^{n} s_k^1 \left(\frac{p_k^0}{p_k^1}\right)^{\frac{r}{2}} \;. \tag{30}$$

and note that (29) becomes

$$\frac{c(p^1)}{c(p^0)} \;=\; \left(\frac{\tilde{\rho}^0}{\tilde{\rho}^1}\right)^{\frac{1}{r}} \;.$$

[17]See Diewert (1976, eq. 4.5, p. 131) and theorem 4.11, p. 133.

6.3. The price curves

Applying lemma 2 allows us to compute the value of the mean order r unit expenditure function along the curve[18]

$$p(t) \;=\; \beta(t) \;=\; \left[(p^0)^{\frac{r}{2}} + t\left((p^1)^{\frac{r}{2}} - (p^0)^{\frac{r}{2}}\right)\right]^{\frac{2}{r}} ,$$

with components

$$p_i(t) \;=\; p_i^0\left[1 \,+\, t\left(\left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}} - 1\right)\right]^{\frac{2}{r}} .$$

Note that $\beta(0) = p^0$ and $\beta(1) = p^1$. The tangent to the β curve at p^0 is

$$\begin{aligned} \bar{v} \;=\; \beta'(0) \;&=\; \frac{2}{r}\left[p_1^0\left(\left(\frac{p_1^1}{p_1^0}\right)^{\frac{r}{2}} - 1\right), \ldots, p_i^0\left(\left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}} - 1\right), \ldots, p_n^0\left(\left(\frac{p_n^1}{p_n^0}\right)^{\frac{r}{2}} - 1\right)\right] \\ &=\; \frac{2}{r}\,\hat{p}^0\left(\left(\hat{p}^0\right)^{-1}p^1\right)^{\frac{r}{2}} - 1\Big) . \end{aligned} \tag{31}$$

The value of the unit expenditure function will not be constant along $\beta(t)$, in fact it will not be so even if $c(p^0) = c(p^1)$ unless we impose severe restrictions on the $[b_{ij}]$ matrix.

We can construct a new curve α along which costs are constant, and thus lies in the $c(p^0)$ factor price frontier, by using the homogeneity of the expenditure function to proportionately adjust all prices, and define[19]

$$\alpha(t) \;= (\tilde{p}_1, \ldots, \tilde{p}_i, \ldots, \tilde{p}_n)(t) \;=\; \frac{c(p^0)}{c(\beta(t))}\,\beta(t) ,$$

the coordinates of the curve being

$$\tilde{p}_i(t) \;=\; \frac{c(p^0)}{c(p(t))}\,p_i(t) \;=\; \frac{c(p^0)}{c(p(t))}\left[(p_i^0)^{\frac{r}{2}} + t\left((p_i^1)^{\frac{r}{2}} - (p_i^0)^{\frac{r}{2}}\right)\right]^{\frac{2}{r}} , \qquad i = 1, \ldots, n .$$

The α curve will not pass through p^1 unless $c(p^0) = c(p^1)$. The initial velocity of the curve $\alpha(t)$ at p^0 $[t = 0]$ is v with components

$$v_i \;=\; \left.\frac{d\,\tilde{p}_i(t)}{d\,t}\right|_{t=0} \;=\; \frac{2}{r}\,p_i^0\left[\left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}} \,-\, \tilde{\rho}^0\right] , \qquad i = 1, \ldots, n. \tag{32}$$

It is readily verified that $v'x = 0$, and thus that $v \in T_{p^0}M$. The presence of the r parameter in (32) indicates that the direction v depends on r!

[18]See (17) for the translog case.

[19]See (19) for the translog case.

6.4. The DSES along the curve

The DSES for the quadratic mean of order r function was computed in (26). We will now evaluate the DSES at p^0 in the direction $v \in T_pM$ determined by the price change between periods 0 an period 1, and given by (32). The numerator in the expression for the DSES is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\left(\frac{p_i^0}{c^0}\frac{p_j^0}{c^0}\right)^{\frac{r}{2}} \frac{v_i}{p_i^0}\frac{v_j}{p_j^0} = \frac{4}{r^2}\left(c^0\right)^{-r}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\left(p_i^0\,p_j^0\right)^{\frac{r}{2}}\left(\left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}} - \tilde{\rho}^0\right)\left(\left(\frac{p_j^1}{p_j^0}\right)^{\frac{r}{2}} - \tilde{\rho}^0\right)$$

$$= \frac{4}{r^2}\left(c^0\right)^{-r}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\left(\left(p_i^1\,p_j^1\right)^{\frac{r}{2}} - \tilde{\rho}^0\left(p_i^1\,p_j^0\right)^{\frac{r}{2}} - \tilde{\rho}^0\left(p_i^0\,p_j^1\right)^{\frac{r}{2}} + \left(\tilde{\rho}^0\right)^2\left(p_i^0\,p_j^0\right)^{\frac{r}{2}}\right)$$

$$= \frac{4}{r^2}\left[\left(\frac{c^1}{c^0}\right)^r - \left(\tilde{\rho}^0\right)^2\right] = \frac{4}{r^2}\left[\frac{\tilde{\rho}^0}{\tilde{\rho}^1} - \left(\tilde{\rho}^0\right)^2\right].$$

The last step follows from (29). Note that we are able able to replace the unknown second order parameters by the observed first order variables. The denominator evaluated at p^0 in the direction (32) is

$$\sum_{i=1}^{n} s_i^0\,\frac{v_i^2}{(p_i^0)^2} = \frac{4}{r^2}\sum_{i=1}^{n} s_i^0\left[\left(\frac{p_i^1}{p_i^0}\right)^r - 2\,\tilde{\rho}^0\left(\frac{p_i^1}{p_i^0}\right)^{\frac{r}{2}} + \left(\tilde{\rho}^0\right)^2\right]$$

$$= \frac{4}{r^2}\left[\sum_{i=1}^{n} s_i^0\left(\frac{p_i^1}{p_i^0}\right)^r - \left(\tilde{\rho}^0\right)^2\right].$$

The DSES at p^0 in the v direction is [see (26)]

$$\mathrm{DSES}_{p^0}(v) = \left(1-\frac{r}{2}\right) - \frac{r}{2}\,\frac{\frac{\tilde{\rho}^0}{\tilde{\rho}^1} - \left(\tilde{\rho}^0\right)^2}{\sum_{i=1}^{n} s_i^0\left(\frac{p_i^1}{p_i^0}\right)^r - \left(\tilde{\rho}^0\right)^2}, \tag{33}$$

with $\tilde{\rho}^0$ and $\tilde{\rho}^1$ given by (30). The expression shows that DSES at p^0 in the direction v of the observed price change is fully determined by the observations (p^0, x^0) and (p^1, x^1), for a given value of the parameter r.[20]

In particular for the Fisher ideal index ($r = 2$) the deflation terms (30) reduce to the Laspeyres index and the inverse of the Paasche index respectively,

$$\tilde{\rho}^0 = \sum_{k=1}^{n} s_k^0\,\frac{p_k^1}{p_k^0} = P^L \qquad \text{and} \qquad \tilde{\rho}^1 = \sum_{k=1}^{n} s_k^1\,\frac{p_k^0}{p_k^1} = \frac{1}{P^P},$$

and the DSES reduces to

$$\mathrm{DSES}_{p^0}(v) = \frac{P^L\left(P^L - P^P\right)}{\sum_{i=1}^{n} s_i^0\left(\frac{p_i^1}{p_i^0} - P^L\right)^2}. \tag{34}$$

[20] It can be shown, as one would expect, that this expression for the DSES converges to the equivalent expression (23) for the translog function as $r \to 0$.

6.5. Quadratic mean illustration

Let us continue with the numerical example introduced in the translog section 5.5 and based on the Norwegian CPI data. Table 3 presents the quadratic mean of order r index for the period 1990–1998 for 5 different values of the parameter r. The index for $r = 0$ is

Table 3: Quadratic mean of order r price indices

	−10	−2	0	2	10
1990	1.000000	1.000000	1.000000	1.000000	1.000000
1991	1.031839	1.032757	1.032784	1.032734	1.031717
1992	1.055537	1.056447	1.056473	1.056421	1.055398
1993	1.078421	1.079747	1.079800	1.079745	1.078396
1994	1.093147	1.094065	1.094117	1.094100	1.093325
1995	1.119238	1.120029	1.120081	1.120075	1.119473
1996	1.132235	1.133104	1.133153	1.133135	1.132397
1997	1.160695	1.161545	1.161587	1.161563	1.160792
1998	1.185922	1.186698	1.186737	1.186715	1.186019

the Törnqvist index already presented in table 1. The values of the indices are so close that I have included six decimals in the presentation.

Of greater interest is table 4, which presents the DSES for the r mean function for five different values of the r parameter. The middle column, for $r = 0$, gives the DSES for the translog function and duplicates the last column in table 1. The value in the

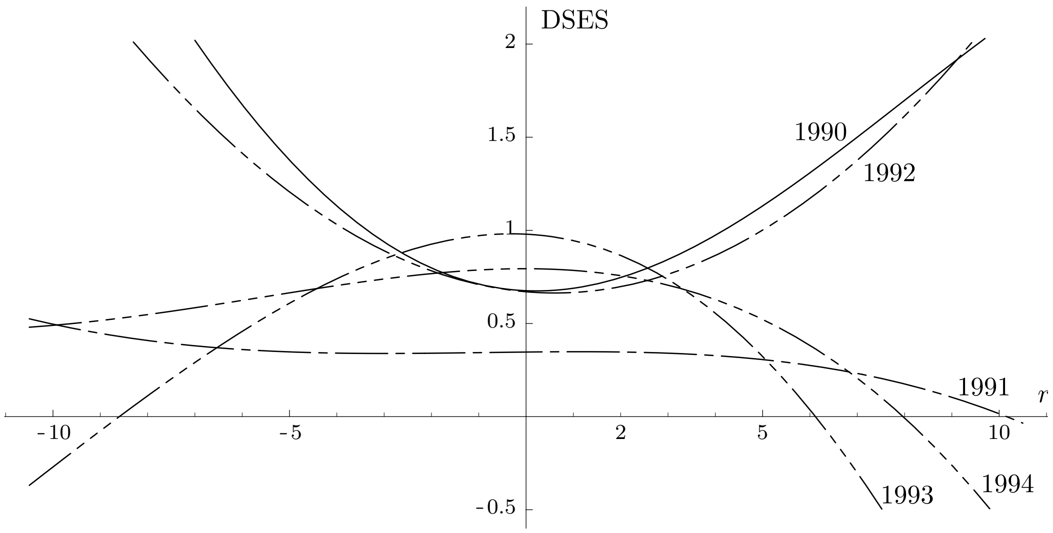
Table 4: The DSES of the quadratic mean of order r price indices

	−10	−2	0	2	10
1990	3.3098	0.7978	0.6760	0.7465	2.0860
1991	0.4962	0.3404	0.3467	0.3469	0.0157
1992	2.5550	0.7823	0.6702	0.6976	2.1875
1993	−0.2705	0.9261	0.9794	0.8653	−1.5978
1994	0.4899	0.7678	0.7941	0.7569	−0.5571
1995	1.9360	1.2783	1.2739	1.3088	1.4115
1996	1.1441	1.0616	1.0808	1.0908	0.8010
1997	1.4219	1.2497	1.2552	1.2368	0.4430

fourth column ($r = 2$) of the first row indicates that the implicit directional elasticity of substitution of Fisher's ideal index in 1990 in the direction of the observed price change between 1990 and 1991 was 0.7465. The values for each year in the three central columns ($r = -2,\ 0,\ 2$) are all rather close, and have the "right" sign. For $r = -10$ or $r = 10$ there is greater variability and some of the DSESs become negative.

This behavior seems to be generic as illustrated by figure 2. Here we have drawn the DSES as a function of r for each of the five years $1990, 1991, \ldots, 1994$.[21] The values in table 4 can be read off the figure. The uppermost, and continuous, curve represents the

Figure 2: The DSES of the quadratic mean of order r price indices, 1990–1994

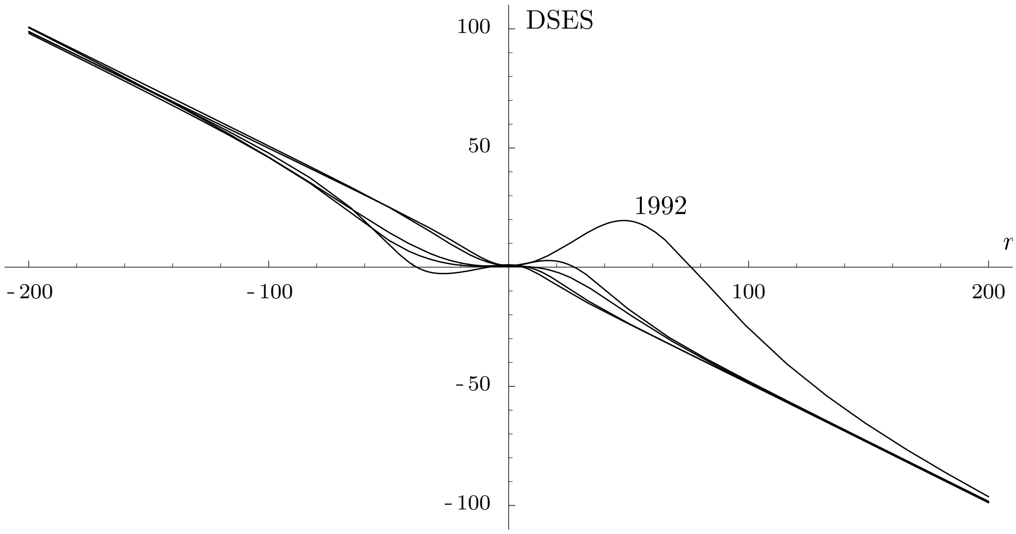


values for 1990, i.e. the DSES of the price change from 1990 to 1991. Each point on the curve represents a different quadratic mean of order r function. The function for 1990 is convex, that for 1993 is concave, while the 1991 curve is "S" shaped and rather flat. All functions are rather flat around 0, and there is in general little difference between the value of the DSES at $r = 0$ (Törnqvist) and $r = 2$ (Fisher's ideal), which confirms the difficulty of choosing between those two functional forms.

As the absolute value of r increases so does that of the DSES, and some of the functions become negative. The local behavior of the curves in the figure is somewhat misleading. It is the case for all 5 curves that the $\text{DSES}(r) \to \infty$ as $r \to -\infty$ and $\text{DSES}(r) \to -\infty$ as $r \to \infty$. The limiting behavior of the DSES as a function of r is illustrated in figure 3 where we have extended the range of the 5 functions illustrated in figure 2. The asymptotes will tend to lie close to the $-0.5\,r$ line, the discrepancy being determined by the value shares associated with the smallest and the largest relative price changes between the periods used to determine the functions.

[21]The remaining years have been left out so as not to overcrowd the figure.

Figure 3: Asymptotic DSES of the quadratic mean of order r price indices, 1990–1994



7. Concluding comments

The paper has demonstrated that if we believe that a superlative price index like the quadratic mean of order r index gives a correct measure of the change in the price level, then it also provides us with sufficient information to measure the curvature of the reference indifference surface in the direction of the observed price change, and in this direction only. The directional shadow elasticity of substitution gives us a theoretically consistent measure of this substitution, and would appear to yield useful quantitative information about the underlying preferences. No other measure of the elasticity of substitution is applicable in this context.

Though it is probably not to be expected that a statistical agency using a superlative CPI would publish the elasticity of substitution implicit in its computation of the index, it would seem to provide an interesting summary statistic for the statisticians themselves. On the other hand the above computations do very little to solve the dilemma between using the Törnqvist or the Fisher index, except perhaps reinforcing the perception that it does not make much difference which one is used.

The DSES may be interpreted as a local measure of the bias and can be used to develop a decomposition of the overall level of that bias into a component which depends on the curvature of the preferences and a component which depends on the magnitude of the price change. We may also use the above analysis to develop a method for extending the superlative indices beyond the sample period somewhat along the lines of Shapiro and Wilcox (1997). It may be noted that the estimates for the DSES in table 1 are fairly close to the $\sigma = 0.7$ of Shapiro and Wilcox (1997) and the procedure may help develop other methods for the computation or real-time superlative indices.

List of symbols

symbol	explanation	page	eqn. nr.
$c(p)$	unit expenditure function	2	
$\mathrm{DSES}_p(v)$	directional shadow elasticity of substitution at p in the direction $v \in T_pM$	8	11
M	price frontier and level surface of c, $\{\, p \mid c(p) \text{ is constant} \,\}$	8	9
$p \in R^n_+$	price vector, $p = (p_1, \ldots, p_n)$	2	
$\tilde{p}$	price on $\alpha(t)$ in M	12	
$P^*(p^1, p^0)$	the true (Konüs) price index	2	
$P(p^1, x^1; p^0, x^0)$	price index	2	
T_pM	tangent plane to M at p	8	10
$v \in T_pM$	price change in the tangent plane T_pM at p	8, 13	11, 21
$\bar{v} \in T_pR^n_+$	price change at p	12	18
x	commodity vector, $x = c_p(p)$	2	
$\alpha(t)$	price curve in M	12	19
$\beta(t)$	price curve in R^n_+	12	17
ρ^0	local deflation factor	13	22
$\tilde{\rho}^\tau$	deflation term, used in r-mean case	19	30
∇f	gradient of f, $\nabla f = (f_1, \ldots, f_n)$,	5	

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