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Matrix oriented formulae of dynamic multipliers in general autoregressive distributed lag and VARX models

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Abstract

Formulae of how to calculate dynamic multipliers in single equation distributed lag models and in VARX models are suggested. An important aspect when constructing the formulae is to represent the models on "companion form". Even if it is not proved that the formulae have general validity, it is demonstrated by examples that they work.

1. Introduction

An important application of open linear dynamic econometric models is to use them to calculate dynamic and cumulative dynamic multipliers. Many textbooks only discuss the most simple case with one lag of the endogenous variable and no lag on the exogenous variable, mentioning that more general cases can be dealt with employing the companion form. In this paper formulae of dynamic multipliers[1] (and then implicitly cumulative dynamic multipliers) of general autoregressive distributed lag models and VARX[2] models which are suitable for matrix-oriented software programs. For all the formulae it is tacitly assumed that the models are stable, cf. Lütkepohl (2005, pp. 18-19). The current paper is inspired by Theil and Boot (1962) who demonstrate in a general way how dynamic systems can be put in companion forms. However they do not provide the formulae given in the current paper. No general proofs of the formulae are given in the current paper, but it is demonstrated that the formulae work in different situations. The software program MAPLE 8 has been heavily utilized for analytical calculations.[3]

The rest of the paper is organized in the following way. In Section 2.1 we consider general autoregressive distributed lag models. Then in sections 2.2 and 2.3 we consider two specific examples of autoregressive distributed lag models. In Section 2.4 we consider a real case looking at the cumulative dynamic multipliers in the export equation of foreigners' consumption in Norway implemented in the quarterly Norwegian macroeconometric model KVARTS. In Section 3 VARX models are considered. Section 3.1 is devoted to general VARX models, whereas a specific VARX model with 3 response variables is discussed in Section 3.2. In Section 4 we comment on some extensions and possible applications.

2.1 ADL-models

Consider the autoregressive distributed lag model

(1) $$A(L)y_t = \mu + B(L)x_t + \varepsilon_t,$$

where L denotes the lag-operator and the 2 lag-polynomials are given by

$$A(L)y_t = 1 - \sum_{i=1}^{k} a_i L^i$$

and

$$B(L) = \sum_{i=0}^{m} b_i L^i.$$

[1] In the literature there are some varying terminology, which may provide confusion. In this paper we apply the terminology employed by Stock and Watson (2007, p. 603). For instance what Stock and Watson refer to as cumulated multipliers Hughes Hallett and Rees (1983, p. 55) speak of as dynamic multipliers. Likewise, what the former authors label dynamic multipliers the latter authors refer to as impact and interim multipliers.

[2] For a discussion of VARX models cf. for instance Reinsel (1997, Ch. 8).

[3] For an introduction to MAPLE cf. Nicolaides and Walkington (1996).

The term ε_t captures all other effects, excluding the effects of other right-hand side variables and noise. The lag length of the left-hand side variable is *k*, whereas that of the specified right-hand side variable is *m*. To distinguish between different members of the class we utilize the notation *ADL*(*k*,*m*). where

We write (1) on the companion form and obtain.

(2) $$y_t^* = Cy_{t-1}^* + Dx_t^* + \varepsilon_t^*$$

The symbols y_t^*, x_t^* and ε_t^* denote column vectors with *f* elements, where $f = \max(k, m+1)$. They are defined as

$$y_t^* = \left[y_t, y_{t-1}, \ldots, y_{t-f+1} \right]',$$

$$x_t^* = \left[x_t, x_{t-1}, \ldots, x_{t-f+1} \right]'$$

and

$$\varepsilon_t^* = [\varepsilon_t, 0, \ldots, 0]'..$$

The specification of the *C* and *D* matrices depend on which argument in the max function that has the highest value. Let us assume that $k = m+1$ (which will be relaxed in the examples below).[4] The matrices *C* and *D* are in this case given by

$$C = \begin{bmatrix} a_1 & a_2 & \ldots & a_{f-1} & a_f \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} b_{10} & b_{11} & \ldots & b_{1,f-2} & b_{1,f-1} \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

[4] If $k < m+1$ the matrix *C* will be augmented by extra columns and the last column will only have zero elements. If $k > m+1$ the matrix *D* will be augmented with extra columns with zeros.

Note that all the elements along the diagonal in C next to the main dialog in the lower triangle equals 1 and that all elements in D except those in the first row are 0. It is claimed that the dynamic multiplier after s periods, denoted G_s, is given by

(3) $$G_s = tr(C^s D);\ s = 0,1,2,\ldots .$$

Below we provide two examples for some evidence that (3) has a general validity.

2.2. The ADL(1,1) case

Let us consider the model

(4) $$y_t = a_1 y_{t-1} + b_{10} x_{1t} + b_{11} x_{1,t-1} + \varepsilon_t,$$

Since (4) is an identity in time, we have

(5) $$y_{t-h} = a_1 y_{t-h-1} + b_{10} x_{1,t-h} + b_{11} x_{1,t-h-1} + \varepsilon_{t-h},\quad h = 1,2,\ldots .$$

Inserting from (5) when $h = 1$ in (4) we obtain

(6) $$y_t = a_1^2 y_{t-2} + b_{10} x_{1t} + \left(b_{11} + a_1 b_{10}\right) x_{1,t-1} + a_1 b_{11} x_{1,t-2} + \varepsilon_t + a_1 \varepsilon_{t-1}.$$

Inserting from (5) when $h = 2$ in (6) yields

(7) $$y_t = a_1^3 y_{t-3} + b_{10} x_{1,t} + \left(b_{11} + a_1 b_{10}\right) x_{1,t-1} + \left(a_1 b_{11} + a_1^2 b_{10}\right) x_{1,t-2} + a_1^2 b_{11} x_{1,t-3} + \varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2}.$$

Inserting from (5) when $h = 3$ in (7) yields

(8) $$\begin{aligned} y_t &= a_1^4 y_{t-4} + b_{10} x_{1t} + \left(b_{11} + a_1 b_{10}\right) x_{1,t-1} + \left(a_1 b_{11} + a_1^2 b_{10}\right) x_{1,t-2} + \left(a_1^2 b_{11} + a_1^3 b_{10}\right) x_{1,t-3} + a_1^3 b_{11} x_{1,t-4} \\ &\quad + z_t + a_1 z_{t-1} + a_1^2 z_{t-2} + a_1^3 z_{t-3}. \end{aligned}$$

Let G_s^* denote dynamic multipliers calculated from (8). It follows directly that

$$G_0^* = b_{10},$$

$$G_1^* = b_{11} + a_1 b_{10},$$

$$G_2^* = a_1 b_{11} + a_1^2 b_{10}$$

and

$$G_3^* = a_1^2 b_{11} + a_1^3 b_{10}.$$

Let us now check what we obtain when we employ (3). In this particular case we have

$$y_t^* = \left(y_t, y_{t-1}\right)',$$

$$x_t^* = (x_t, x_{t-1})',$$

$$\varepsilon_t^* = (\varepsilon_t, \varepsilon_{t-1})',$$

$$C = \begin{bmatrix} a_1 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} b_{10} & b_{11} \\ 0 & 0 \end{bmatrix}$$

It now follows from (3) that

$$G_0 = tr(D) = b_{10},$$

$$G_1 = tr(CD) = tr\left[\begin{pmatrix} a_1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_{10} & b_{11} \\ 0 & 0 \end{pmatrix}\right] = tr\begin{bmatrix} a_1 b_{10} & a_1 b_{11} \\ b_{10} & b_{11} \end{bmatrix} = a_1 b_{10} + b_{11},$$

$$G_2 = tr(C^2 D) = tr\left[\begin{pmatrix} a_1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 b_{10} & a_1 b_{11} \\ b_{10} & b_{11} \end{pmatrix}\right] = tr\begin{bmatrix} a_1^2 b_{10} & a_1^2 b_{11} \\ a_1 b_{10} & a_1 b_{11} \end{bmatrix} = a_1^2 b_{10} + a_1 b_{11},$$

$$G_3 = tr(C^3 D) = tr\left[\begin{pmatrix} a_1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1^2 b_{10} & a_1^2 b_{11} \\ a_1 b_{10} & a_1 b_{11} \end{pmatrix}\right] = tr\begin{bmatrix} a_1^3 b_{10} & a_1^3 b_{11} \\ a_1^2 b_{10} & a_1^2 b_{11} \end{bmatrix} = a_1^3 b_{10} + a_1^2 b_{11}.$$

We notice that $G_s = G_s^*;\ s = 0,1,2,3.$

2.3. The ADL(2,2) model

Let us now consider the somewhat more general model

(9) $$y_t = a_1 y_{t-1} + a_2 y_{t-2} + b_{10} x_{1t} + b_{11} x_{1,t-1} + b_{12} x_{1,t-2} + \varepsilon_t.$$

Since (9) is an identity in time we have

(10) $$y_{t-h} = a_1 y_{t-h-1} + a_2 y_{t-h-2} + b_{10} x_{1,t-h} + b_{11} x_{1,t-h-1} + b_{12} x_{1,t-h-2} + \varepsilon_{t-h},\quad h = 1,2,3,\dots .$$

Inserting from (10) when $h = 1$ into (9) we obtain

(11) $$\begin{aligned} y_t &= (a_1^2 + a_2) y_{t-2} + a_1 a_2 y_{t-3} \\ &\quad + b_{10} x_t + (a_1 b_{10} + b_{11}) x_{1,t-1} + (a_1 b_{11} + b_{12}) x_{1,t-2} + a_1 b_{12} x_{1,t-3} + \varepsilon_t + a_1 \varepsilon_{t-1}. \end{aligned}$$

Inserting from (10) when $h = 2$ yields

$$
\begin{aligned}
y_t &= \left(a_1^3 + 2a_1a_2\right) y_{t-3} + \left(a_1^2 a_2 + a_2^2\right) y_{t-4} + b_{10}x_{1t} + \left(a_1b_{10} + b_{11}\right) x_{1,t-1} \\
&\quad + \left(a_1^2 b_{10} + a_2b_{10} + a_1b_{11} + b_{12}\right) x_{1,t-2} + \left(a_1^2 b_{11} + a_2b_{11} + a_1b_{12}\right) x_{1,t-3} + \left(a_1^2 b_{12} + a_2b_{12}\right) x_{1,t-4} \\
&\quad + \varepsilon_t + a_1\varepsilon_{t-1} + \left(a_1^2 + a_2\right)\varepsilon_{t-2}.
\end{aligned} \tag{12}
$$

Inserting from (10) when $h = 3$ in (12) yields

$$
\begin{aligned}
y_t &= \left(a_1^4 + 3a_1^2a_2 + a_2^2\right) y_{t-4} + \left(a_1^3 a_2 + 2a_1a_2^2\right) y_{t-5} + b_{10}x_{1t} + \left(a_1b_{10} + b_{11}\right) x_{1,t-1} \\
&\quad + \left(a_1^2 b_{10} + a_2b_{10} + a_1b_{11} + b_{12}\right) x_{1,t-2} + \left(a_1^2 b_{11} + a_2b_{11} + a_1b_{12} + a_1^3b_{10} + 2a_1a_2b_{10}\right) x_{1,t-3} \\
&\quad + \left(a_1^2 b_{12} + a_2b_{12} + a_1^3b_{11} + 2a_1a_2b_{11}\right) x_{1,t-4} + \left(a_1^3 b_{12} + 2a_1a_2b_{12}\right) x_{1,t-5} \\
&\quad + \varepsilon_t + a_1\varepsilon_{t-1} + \left(a_1^2 + a_2\right)\varepsilon_{t-2} + \left(a_1^3 + 2a_1a_2\right)\varepsilon_{t-3}.
\end{aligned} \tag{13}
$$

In this case we have

$$y_t^* = \left[y_t, y_{t-1}, y_{t-2}\right]',$$

$$x_t^* = \left[x_t, x_{t-1}, x_{t-2}\right]',$$

$$\varepsilon_t^* = \left[\varepsilon_t, 0, 0\right]',$$

$$C = \begin{bmatrix} a_1 & a_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It now follows from (13) that

$$G_0 = b_{10},$$

$$G_1 = b_{11} + a_1b_{10},$$

$$G_2 = a_1^2 b_{10} + a_2b_{10} + a_1b_{11} + b_{12}$$

and

$$G_3 = a_1^2 b_{11} + a_2b_{11} + a_1b_{12} + a_1^3b_{10} + 2a_1a_2b_{10}.$$

Likewise it follows from (3) that

$$G_0^* = tr(D) = b_{10},$$

$$G_1^* = tr(CD) = tr\left[\begin{pmatrix} a_1 & a_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = tr\begin{pmatrix} a_1 b_{10} & a_1 b_{11} & a_1 b_{12} \\ b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 \end{pmatrix} = a_1 b_{10} + b_{11},$$

$$G_2^* = tr(C^2 D) = tr\left[\begin{pmatrix} a_1 & a_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 b_{10} & a_1 b_{11} & a_1 b_{12} \\ b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 \end{pmatrix}\right]$$
$$= tr\begin{pmatrix} a_1^2 b_{10} + a_2 b_{10} & a_1^2 b_{11} + a_2 b_{11} & a_1^2 b_{12} + a_2 b_{12} \\ a_1 b_{10} & a_1 b_{11} & a_1 b_{12} \\ b_{10} & b_{11} & b_{12} \end{pmatrix} = a_1^2 b_{10} + a_2 b_{10} + a_1 b_{11} + b_{12}$$

and

$$G_3^* = tr(C^3 D) = tr\left[\begin{pmatrix} a_1 & a_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1^2 b_{10} + a_2 b_{10} & a_1^2 b_{11} + a_2 b_{11} & a_1^2 b_{12} + a_2 b_{12} \\ a_1 b_{10} & a_1 b_{11} & a_1 b_{12} \\ b_{10} & b_{11} & b_{12} \end{pmatrix}\right]$$
$$= tr\begin{pmatrix} a_1^3 b_{10} + 2a_1 a_2 b_{10} & a_1^3 b_{11} + 2a_1 a_2 b_{11} & a_1^3 b_{12} + 2a_1 a_2 b_{12} \\ a_1^2 b_{10} + a_2 b_{10} & a_1^2 b_{11} + a_2 b_{11} & a_1^2 b_{12} + a_2 b_{12} \\ a_1 b_{10} & a_1 b_{11} & a_1 b_{12} \end{pmatrix}$$
$$= a_1^3 b_{10} + 2a_1 a_2 b_{10} + a_1^2 b_{11} + a_2 b_{11} + a_1 b_{12}.$$

Again we notice that $G_s = G_s^*;\ \ s = 0,1,2,3.$

2.4 An empirical example

Consider the following example taken from the export relation in the Norwegian macroeconometric model KVARTS related to foreigners' consumption in Norway. The following error correction model is employed

(14) $$y_t - y_{t-1} = \alpha_1 y_{t-1} + \alpha_2\left(y_{t-1} - y_{t-4}\right) + \beta_2^*\left(x_{1,t} - x_{1,t-1}\right) + \beta_1 x_{1,t-1} + \gamma_1^* x_{2,t-3} + \gamma_2\left(x_{2,t-1} - x_{2,t-3}\right) + z_t,$$

where y denotes the log of foreigner's consumption in Norway in constant prices, x_1 denotes the log of a indicator of the economic activity abroad and x_2 denotes the log of the price index of foreigner's consumption in Norway divided by the price index of Norwegian consumption abroad. All other terms, including an error term is included in the scalar z_t.

If we write (14) as an autoregressive distributed lag-model we obtain

(15) $$y_t = a_1 y_{t-1} + a_4 y_{t-4} + b_0 x_{1,t} + b_1 x_{1,t-1} + c_1 x_{2,t-1} + c_3 x_{2,t-3} + z_t ,$$

where

$$a_1 = 1 + \alpha_1 + \alpha_2 ,$$
$$a_4 = -\alpha_2 ,$$
$$b_0 = \beta_2 ,$$
$$b_1 = \beta_1 - \beta_2 ,$$
$$c_1 = \gamma_2 ,$$

and

$$c_3 = \gamma_1 - \gamma_2 .$$

We may write (15) in the two following equivalent ways

(16) $$y_t = a_1 y_{t-1} + a_4 y_{t-4} + b_0 x_{1,t} + b_1 x_{1,t-1} + z_{1,t}$$

and

(17) $$y_t = a_1 y_{t-1} + a_4 y_{t-4} + c_1 x_{2,t-1} + c_3 x_{2,t-3} + z_{2,t} ,$$

where

$$z_{1,t} = c_1 x_{2,t-1} + c_3 x_{2,t-3} + z_t$$

and

$$z_{2,t} = b_0 x_{1,t} + b_1 x_{1,t-1} + z_t .$$

If we write (16) and (17) on companion forms [corresponding to (2)] we have, respectively,

$$y_t^* = C y_{t-1}^* + D_1 x_t^* + z_{1,t}^*$$

and

$$y_t^* = C y_{t-1}^* + D_2 x_t^* + z_{2,t}^* ,$$

where

$$y_t^* = \left[y_t , y_{t-1} , y_{t-2} , y_{t-3} \right]' ,$$

$$x_t^* = \left[x_t , x_{t-1} , x_{t-2} , x_{t-3} \right]' ,$$

$$z_{1t}^* = \left[z_{1t} , 0, 0, 0 \right]' ,$$

$$z_{2t}^{*} = [z_{2t}, 0, 0, 0]',$$

$$C = \begin{bmatrix} a_1 & 0 & 0 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} b_0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$D_2 = \begin{bmatrix} 0 & c_1 & 0 & c_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The (estimated) values of the parameters in (52) are $\hat{a}_1 = 0.2066$, $\hat{a}_4 = 0.3875$, $\hat{b}_0 = 0.5488$ and $\hat{b}_1 = 0.276$, $\hat{c}_1 = 0.9048$ and $\hat{c}_3 = 0.7322$.

The cumulative dynamic multipliers related to x_1 and x_2 are now given by

$$CG_{is} = tr(C^s D_i);\ s = 0,1,2,\ldots .$$

The numerical results for the 40 first periods are reported in Table 1.

Table 1. Cumulative dynamic multipliers related to Eq. (15)

Cumulative dynamic multipliers after different number of quarters	x_1	x_2
0	0.5488	0
1	0.3862	-0.9048
2	0.3526	-1.0917
3	0.3456	-0.3982
4	0.5569	-0.2549
5	0.5375	-0.5759
6	0.5205	-0.7146
7	0.5143	-0.4745
8	0.5948	-0.3694
9	0.6040	-0.4721
10	0.5993	-0.5470
11	0.5959	-0.4695
12	0.6264	-0.4127
13	0.6363	-0.4408
14	0.6365	-0.4756
15	0.6352	-0.4528
16	0.6468	-0.4261
17	0.6530	-0.4314
18	0.6543	-0.4460
19	0.6541	-0.4402
20	0.6586	-0.4287
21	0.6619	-0.4283
22	0.6631	-0.4339
23	0.6633	-0.4328
24	0.6650	-0.4281
25	0.6667	-0.4270
26	0.6675	-0.4290
27	0.6677	-0.4289
28	0.6684	-0.4271
29	0.6692	-0.4263
30	0.6697	-0.4269
31	0.6699	-0.4270
32	0.6702	-0.4263
33	0.6706	-0.4259
34	0.6709	-0.4260
35	0.6710	-0.4261
36	0.6711	-0.4258
37	0.6713	-0.4256
38	0.6715	-0.4256
39	0.6715	-0.4256
40	0.6716	-0.4255

3. VARX models

3.1 VARX models. General remarks

Consider the following VARX model

(18) $$y_t = \sum_{i=1}^{k} A_i y_{t-i} + \sum_{j=0}^{m} B_j x_{t-j} + \varepsilon_t ,$$

where y_t is a p-dimensional column-vector defined by

$$y_t = \left(y_{1t}, y_{2t}, ..., y_{pt} \right)' ,$$

A_i is a p×p-matrix defined by

$$A_i = \begin{bmatrix} a_{11i} & a_{12i} & \cdots & a_{1pi} \\ a_{21i} & a_{22i} & \cdots & a_{2pi} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1i} & a_{p2i} & \cdots & a_{ppi} \end{bmatrix}; \; i = 1, 2, ..., k ,$$

x_t is the scalar exogenous variable of primary interest, B_j is a p×1 column vector given by

$$B_j = \left[B_{1j}, B_{2j}, ..., B_{pj} \right]'; \; j = 0, 1, ..., m$$

and ε_t is a vector covering all other terms, for instance other exogenous variables and error terms. Below we will consider the case when $k = m + 1$. Let $\tilde{a}_{qr}$ and $\tilde{a}_{qq}$ be k×k matrices defined by

$$\tilde{a}_{qr} = \begin{bmatrix} a_{qr1} & a_{qr2} & \cdots & a_{qrk} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}; \; q, r = 1, ..., p, \; q \neq r$$

and

$$\tilde{a}_{qq} = \begin{bmatrix} a_{qq1} & a_{qq2} & \cdots & a_{qq,k-1} & a_{qq,k} \\ 1 & 0 & \vdots & 0 & 0 \\ 0 & 1 & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; q = 1, ..., p.$$

Let furthermore the k×k matrix $\tilde{b}_q$ be defined by

$$\tilde{b}_q = \begin{bmatrix} b_{q0} & b_{q1} & \cdots & b_{q,k-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix};\ q = 1,\dots,p$$

and let $\tilde{d}_q$ be a k×p matrix with zeros everywhere except in the q'th column of the first row ($q = 1,\dots,p$).

We also define

$$y^*_{qt} = \left[y_{q,t}, y_{q,t-1}, \dots, y_{q,t-(k-1)} \right]',\ q = 1,\dots,p,$$

$$y^*_t = \left[y^{*\prime}_{1,t}, y^{*\prime}_{2,t}, \dots, y^{*\prime}_{p,t} \right]'$$

and

$$x^*_t = \left[x_t, x_{t-1}, \dots, x_{t-(k-1)} \right]'.$$

Let furthermore the matrices *A*, *B* and *D* be defined as

$$A = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1p} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{p1} & \tilde{a}_{p2} & \cdots & \tilde{a}_{pp} \end{bmatrix},$$

$$B = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_p \end{bmatrix} \quad \text{and}$$

$$D = \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \\ \vdots \\ \tilde{d}_p \end{bmatrix}.$$

We may now represent (18) as

$$y^*_t = Ay^*_{t-1} + Bx^*_t + D\varepsilon_t.$$

Note that the matrices *A*, *B* and *D* are of dimension (*pk*)×(*pk*), (*pk*)×*k* and (*pk*)×*p*, respectively. Let R_q be a partitioned *k*×*pk* matrix, which consists of p quadratic *k*×*k* matrices. All the quadratic submatrices but the q'th are zero matrices. The q'th submatrix is the identity matrix. The dynamic multipliers of y_q is now given by

$$(19) \qquad G_{qs}^{*} = tr\left(R_q A^s B\right);\ q = 1,...,p;\ s = 0,1,2,....$$

3.2 VARX models. Specific case

Consider now the specific case represented by the following VARX-model where we assume that $p = 3$.

$$(20) \qquad y_t = A_1 y_{t-1} + A_2 y_{t-2} + B_0 x_t + B_1 x_{t-1} + B_2 x_{t-2} + \varepsilon_t .$$

Since (20) is an identity in time, it also follows that

$$(21) \qquad y_{t-h} = A_1 y_{t-h-1} + A_2 y_{t-h-2} + B_0 x_{t-h} + B_1 x_{t-h-1} + B_2 x_{t-h-2} + \varepsilon_{t-h},\ h = 1,2,....$$

Inserting from (21) when $h = 1$ and $h = 2$ in (20) yields

$$(22) \qquad \begin{aligned} y_t = {} & A_1^2 y_{t-2} + \left(A_1 A_2 + A_2 A_1\right) y_{t-3} + A_2 A_2 y_{t-4} + B_0 x_t + \left(A_1 B_0 + B_1\right) x_{t-1} + \left(A_1 B_1 + A_2 B_0 + B_2\right) x_{t-2} \\ & + \left(A_1 B_2 + A_2 B_1\right) x_{t-3} + A_2 B_2 x_{t-4} + \varepsilon_t + A_1 \varepsilon_{t-1}. \end{aligned}$$

Inserting from (21) when $h = 2$ in (22) yields

$$(23) \qquad \begin{aligned} y_t = {} & \left(A_1 A_2 + A_2 A_1 + A_1^3\right) y_{t-3} + \left(A_2^2 + A_1^2 A_2\right) y_{t-4} + B_0 x_t + \left(A_1 B_0 + B_1\right) x_{t-1} \\ & + \left(A_1 B_1 + A_2 B_0 + B_2 + A_1^2 B_0\right) x_{t-2} + \left(A_1 B_2 + A_2 B_1 + A_1^2 B_1\right) x_{t-3} + \left(A_2 B_2 + A_1^2 B_2\right) x_{t-4} \\ & + \varepsilon_t + A_1 \varepsilon_{t-1} + \left(A_1^2 + A_2\right) \varepsilon_{t-2}. \end{aligned}$$

Let

$$A_i = \begin{bmatrix} a_{11i} & a_{12i} & a_{13i} \\ a_{21i} & a_{22i} & a_{23i} \\ a_{31i} & a_{32i} & a_{33i} \end{bmatrix};\ i = 1,2$$

and

$$B_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ b_{3i} \end{bmatrix}, i = 0,1,2.$$

The *A* and *B* matrices in this case have the forms

$$A = \begin{bmatrix} a_{111} & a_{112} & 0 & a_{121} & a_{122} & 0 & a_{131} & a_{132} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{211} & a_{212} & 0 & a_{221} & a_{222} & 0 & a_{231} & a_{232} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_{311} & a_{312} & 0 & a_{321} & a_{322} & 0 & a_{331} & a_{332} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{20} & b_{21} & b_{22} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{30} & b_{31} & b_{32} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The selection matrix D is given by

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore we define

$$y_t^* = \left[y_{1,t}, y_{1,t-1}, y_{1,t-2}, y_{2,t}, y_{2,t-1}, y_{2,t-2}, y_{3,t}, y_{3,t-1}, y_{3,t-2} \right]'$$

and

$$x_t^* = \left[x_t, x_{t-1}, x_{t-2} \right]'.$$

Let G_{1s} ($s = 0,1,2$) denote the calculated dynamic multipliers for y_1 based on Equation (23). Let $\iota_3 = (1,0,0)'$. It then follows that

$$G_{10} = \iota_3' B_0 = b_{10},$$

$$G_{11} = \iota_3' A_1 B_0 + \iota_3' B_1 = a_{111} b_{10} + a_{121} b_{20} + a_{131} b_{30} + b_{11}$$

and

$$\begin{aligned} G_{12} = \iota_3' A_1 B_1 + \iota_3' A_2 B_0 + \iota_3' B_2 + \iota_3' A_1^2 B_0 = a_{111} b_{11} + a_{121} b_{21} + a_{131} b_{31} + a_{112} b_{10} + a_{122} b_{20} + a_{132} b_{30} + b_{12} \\ + a_{111}^2 b_{10} + a_{121} a_{211} b_{10} + a_{131} a_{311} b_{10} + a_{111} a_{121} b_{20} + a_{121} a_{221} b_{20} + a_{131} a_{321} b_{20} \\ + a_{111} a_{131} b_{30} + a_{121} a_{231} b_{30} + a_{131} a_{331} b_{30}. \end{aligned}$$

Employing Eq. (19) yields

$$G_{10}^* = b_{10},$$

$$G_{11}^* = a_{111} b_{10} + a_{121} b_{20} + a_{131} b_{30} + b_{11}$$

and

$$\begin{aligned} G_{12}^* = a_{111}^2 b_{10} + a_{111} a_{121} b_{20} + a_{111} a_{131} b_{30} + a_{112} b_{10} + a_{121} a_{211} b_{10} + a_{121} a_{221} b_{20} + a_{121} a_{231} b_{30} + a_{122} b_{20} \\ + a_{131} a_{311} b_{10} + a_{131} a_{321} b_{20} + a_{131} a_{331} b_{30} + a_{111} b_{11} + a_{121} b_{21} + a_{131} b_{31} + b_{12}. \end{aligned}$$

We note that

$$G_{1j}^* = G_{1j}; \; j = 0,1,2.$$

4. Extensions and applications

The formulae given in Section 3.1 is of course also relevant for small linear interdependent dynamic systems even if the current paper does not contain any example for such type of models. The reduced form of such systems can easily be derived by multiplying the vector equation by the inverse of a matrix containing the contemporary effects. When this is done the formulae in Section 3.1 can be employed. In dynamic econometrics a main focus is on the magnitude of the responses of endogenous variables to changes in exogenous variables and on the speed of adjustment. Dynamic multipliers and cumulative dynamic multipliers involve a lot of parameters which enter in a rather non-linear fashion. Even if the models are correctly specified these measures will be uncertain because of parameter uncertainty. Thus it is relevant to ask how precisely these measures are estimated. Even if there are some analytical results (cf. for instance Schmidt, 1973) it may be easier to utilize different bootstrap techniques to obtain for instance estimates of standard errors of dynamic multipliers and cumulative dynamic multipliers. In this case the formulae presented in this paper could be utilized.

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Appendix

A TSP-program[5] (version 5) related to the second column of Table 1

```
options crt;
name Cumulative dynamic multipliers;
? Foreigners' consumption in Norway
? Initialization of the matrices C and D
mform(NCOL=4,NROW=4) C;
mform(NCOL=4,NROW=4) D;
set alfa1=-0.4059;
set alfa2=-0.3875;
set beta2=0.5488;
set beta1=0.2728;
set a1=1+alfa1+alfa2;
set a4=-alfa2;
set b0=beta2;
set b1=beta1-beta2;
? Assigning values to the matrices C and D
set C[1,1]=a1;
set C[1,4]=a4;
set C[2,1]=1;
set C[3,2]=1;
set C[4,3]=1;

set D[1,1]=b0;
set D[1,2]=b1;

mat FF=D;
mat GG=D;

mat rini1=tr(K);
set rini=rini1;

mform(NCOL=1,NROW=41) Z=0;
set z[1,1]=rini;

do k=1 to 40 by 1;
set j=k+1;
mat FF=C*FF;
mat GG=GG+FF;
mat r=tr(GG);
set z[j,1]=r;
enddo;

smpl 1 41;
mmake z2 z;
write(file='mi.txt',format='(1F10.4)') z2;
```

[5] Cf. Hall and Cummins (2005).

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