

Statistics Norway's Open Research Repository - *SNORRe*

<https://brage.bibsys.no/xmlui/handle/11250/177677>

This is a post-peer-review, pre-copyedit version of an article published in *Journal of Optimization theory and application*. The final authenticated version is available online at <https://doi.org/10.1007/s10957-012-0061-2>

Brasch, T., Byström, J. & Lystad, L. P. (2012). Optimal Control and the Fibonacci Sequence. *Journal of Optimization Theory and Applications*, 154(3), 857-878. <https://doi.org/10.1007/s10957-012-0061-2>

This file was downloaded from the Institutional repository at Statistics Norway (SNORRe). This is the final text version of the article after peer review, and it may contain minor differences from the pdf-version.

Dette er siste tekstversjon av artikkelen etter fagfelle vurderingen, og den kan inneholde små forskjeller fra forlagets PDF-versjon.

Optimal Control and the Fibonacci Sequence*

Thomas von Brasch[†] Johan Byström[‡] Lars Petter Lystad[§]

Communicated by F. Giannessi

Abstract

We bridge mathematical number theory with that of optimal control and show that a generalised Fibonacci sequence enters the control function of finite horizon dynamic optimisation problems with one state and one control variable. In particular, we show that the recursive expression describing the first-order approximation of the control function can be written in terms of a generalised Fibonacci sequence, when restricting the final state to equal the steady-state of the system. Further, by deriving the solution to this sequence, we are able to write the first-order approximation of optimal control explicitly. Our procedure is illustrated in an example often referred to as the Brock-Mirman economic growth model.

Keywords: Brock-Mirman model, Fibonacci sequence, Golden ratio, Mathematical number theory, Optimal control.

AMS: 11B39, 93C55, 37N40, 49N10.

1 Introduction

Approximating optimal control problems has a long history and dates at least back to McReynolds ?. Lystad ? and Magill ?? are early applications of the first-order approximation within economics. A good account of how the technique has been used in economics can be found in Judd ?. Recently, this method of approximation has been extended to also handle stochastic rational expectation models with forward looking variables; see, e.g., Levine et al. ? and Benigno and Woodford ?.

The field of bridging optimal control and number theory via the Fibonacci sequence is relatively new. Benavoli et al. ? show the relationship between the Fibonacci sequence and the Kalman filter with a

*Thanks to three anonymous reviewers, Ådne Cappelen, John Dagsvik, Pål Boug and Anders Rygh Swensen for useful comments. The usual disclaimer applies.

[†]Ph.D. student, Statistics Norway: Research department, Unit for macroeconomics, NO-0033 Oslo, Norway and NUI: Department of International Economics, NO-0033 Oslo, Norway. E-mail: vonbrasch@gmail.com (Corresponding author)

[‡]Associate Professor, Luleå University of Technology: Department of Mathematics, SE-97187 Luleå, Sweden. E-mail: johanb@ltu.se

[§]Professor, Narvik University College: Department of Technology, NO-8505 Narvik, Norway. E-mail: lpl@hin.no

simple structure of the plant model. Capponi et al. ? derive a similar result in a continuous time setting. Donoghue ? shows a linkage between the Kalman filter, the linear quadratic control problem and a Fibonacci system defined by adding a control input to the recursion relation generating the Fibonacci numbers. Byström et al. ? derive a relationship between linear quadratic problems and a generalised Fibonacci sequence. We build upon and extend these results for control problems in a generalised form.

The main contribution of this article is to bridge the area of mathematical number theory with that of optimal control. This is done by using a generalised Fibonacci sequence for solving finite horizon dynamic optimisation problems with one state and one control variable. The solution method proposed reveals important properties of the optimal control problem. In particular, we show how the first-order approximation of the optimal control function can be written in terms of these generalised Fibonacci numbers. Further, by developing the explicit solution of the generalised Fibonacci sequence, we are able to provide a non-recursive solution of the first-order approximation.

The structure of the paper is as follows. In Section ??, the optimal control problem is defined and the expression describing the first-order approximation of the optimal control is stated. In Section ??, we contribute to the literature by developing the linkage between the optimal control function and the Fibonacci sequence. To illustrate our procedure, we show how the method can be applied to the Brock-Mirman economic growth model. We derive explicit solutions to the generalised Fibonacci numbers in Section ??, which further enables us to write the first-order approximation of optimal control explicitly. The last section contains a summary and concluding remarks.

2 The Optimal Control Problem

The deterministic optimal control problem consists of minimising an objective function subject to the process describing the evolution of the state variable, given a restriction on the terminal state variable.¹ For $0 \leq t \leq T - 1$, we define the objective function

$$\sum_{t=0}^{T-1} \beta^t f(x_t, u_t), \quad (1)$$

where $0 < \beta \leq 1$ is a discount factor, $x_t \in \mathbb{R}$ represents a state variable and $u_t \in \mathbb{R}$ denotes the control variable. Further, it is assumed that standard regularity conditions hold, i.e., the criterion function f is sufficiently smooth and convex and policies, that are feasible, lie within a compact and convex set. More specific, we will from now on assume that f is twice differentiable and that the Hessian of f is positive

¹The optimal control problem has been widely used within the field of economics; see e.g., Ljungqvist and Sargent ?.

definite. The evolution of the state variable is described by the discrete time system

$$x_{t+1} := Ax_t + Bu_t, \quad t = 0, 1, \dots, T - 1, \quad (2)$$

for a given initial condition x_0 . We assume the existence of a control, which ensures that the state never changes. We refer to such a control as a steady-state control and denote it \bar{u} and, correspondingly, we denote the steady-state \bar{x} . The final state of the discrete time system (??) is restricted to be the steady-state

$$x_T = \bar{x}. \quad (3)$$

From this it follows that a steady-state is characterised by two properties. First, the state is constant and thus time invariant. Second, the steady-state control is optimal, i.e., given that the system starts out at the steady-state, it is optimal to remain at the steady-state through all time periods. The assumption that there exists a steady-state is both necessary and sufficient in order to use the generalised Fibonacci sequence to write the first-order approximation of optimal control explicitly.

The optimal control problem is therefore, the problem of minimising the objective function (??) subject to both the transition function (??) and the fixed final state (??).

Even though the optimal control problem is deterministic, the approach used in this article can be generalised to handle stochastic control problems; see, e.g., Levine et al. ? and Benigno and Woodford ?.

In general, it is not possible to find an explicit expression describing the optimal control function. However, it may be possible to find a recursive expression describing the first-order approximation of the optimal control. In the following well known result, we let the second partial derivatives of the criterion function f , evaluated at the steady-state, be denoted by $f_{\bar{x}\bar{x}} := \frac{\partial^2 f}{\partial \bar{x} \partial \bar{x}}$, $f_{\bar{x}\bar{u}} := \frac{\partial^2 f}{\partial \bar{x} \partial \bar{u}}$ and $f_{\bar{u}\bar{u}} := \frac{\partial^2 f}{\partial \bar{u} \partial \bar{u}}$.

Theorem 2.1 *Consider the optimal control problem, i.e., minimising (??) subject to (??) and (??). The first-order approximation is given by the linear control function (for $0 \leq t \leq T - 1$)*

$$u_t = \bar{u} - (L_t^{\mathbf{a}} - L_t^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}})(x_t - \bar{x}), \quad (4)$$

where $L_t^{\mathbf{a}}$ is given by the equations

$$L_t^{\mathbf{a}} := (f_{\bar{u}\bar{u}} + \tilde{B}S_{t+1}\tilde{B})^{-1}(\tilde{B}S_{t+1}\tilde{A}), \quad (5)$$

$$S_t = \tilde{A}^2 S_{t+1} f_{\bar{u}\bar{u}} (f_{\bar{u}\bar{u}} + \tilde{B}^2 S_{t+1})^{-1} + \tilde{R}, \quad S_T = 0, \quad (6)$$

where the second equation is known as the Riccati equation and where we have used the auxiliary variables $\tilde{A} := \beta^{1/2}(A - Bf_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$, $\tilde{B} := \beta^{1/2}B$ and $\tilde{R} := (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$. The last part of the feedback coefficient ($L_t^{\mathbf{b}}$) represents the linear part which ensures that the control function will drive the state to the steady-state at the final time period

$$L_t^{\mathbf{b}} := (f_{\bar{u}\bar{u}} + \tilde{B}S_{t+1}\tilde{B})^{-1}\tilde{B}W_{t+1}P_t^{-1}W_t, \quad (7)$$

where the two auxiliary variables W_t and P_t are given by

$$W_t = (\tilde{A} - \tilde{B}L_t^{\mathbf{a}})W_{t+1}, \quad W_T = 1, \quad (8)$$

$$P_t = P_{t+1} - W_{t+1}^2\tilde{B}^2(f_{\bar{u}\bar{u}} + \tilde{B}^2S_{t+1})^{-1}, \quad P_T = 0. \quad (9)$$

Proof See Appendix ??.

□

3 Connecting Fibonacci with Optimal Control

The Fibonacci sequence is named after the Italian mathematician Leonardo Pisano Bigollo (1170 - c. 1250), most commonly known as Leonardo Fibonacci. With his most important work, the book of number theory, Liber Abaci, he spread the Hindu-Arabic numeral system to Europe. In Liber Abaci, he also introduced what many will associate him with today, the Fibonacci sequence

$$\{F_n\}_{n=0}^{\infty} = 0, 1, 1, 2, 3, 5, 8, 13, \dots$$

This sequence is characterised by the initial values 0 and 1 and each subsequent number being the sum of the previous two. It is thus fully described by the difference equation

$$F_n = F_{n-1} + F_{n-2},$$

with initial values $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been connected to such diverse fields as nature, art, geometry, architecture, music and even for the calculation of π ; see, e.g., Castellanos ?. One of the most fascinating facts is that the ratio of two consecutive numbers ($H_n := F_{n-1}/F_n$)

$$\{H_n\} = 0, \quad 1, \quad .5, \quad .666\dots, \quad .600, \quad .625, \quad .615\dots, \quad .619\dots, \quad .617\dots, \quad .618\dots, \quad \dots \quad (10)$$

converges to the inverse of the golden ratio: $\phi^{-1} := 2/(1 + \sqrt{5}) \approx .618$. The golden ratio is mathematically interesting for a variety of reasons, e.g., it holds the property that its square is equal to $\phi + 1$ and its inverse is equal to the number itself minus one, i.e.,

$$\phi^{-1} = \phi - 1. \quad (11)$$

The main contribution of this article consists in connecting a generalised Fibonacci sequence (sometimes also denoted Lucas sequence) to optimal control theory.

Definition 3.1 (Generalised Fibonacci sequence) The generalised Fibonacci sequence is defined by the second-order difference equation

$$\mathcal{F}_{n+2} = a\mathcal{F}_{n+1} + b_{n+2}\mathcal{F}_n, \quad (12)$$

with the constant coefficient a , the time varying coefficient b_{n+2} and with given initial values $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$.

Moreover, we define the ratio of two consecutive generalised Fibonacci numbers by

$$\mathcal{H}_n := \mathcal{F}_{n-1}/\mathcal{F}_n, \quad n = 1, 2, \dots$$

Theorem 3.1 Consider the generalised Fibonacci sequence with the particular coefficients $a = \beta^{1/2}B$ and $b_{n+2} = f_{\bar{u}\bar{u}}\tilde{R}^{-1}\tilde{A}^2$, when n is even, and $b_{n+2} = f_{\bar{u}\bar{u}}\tilde{R}^{-1}$, when n is odd, where we have used the auxiliary transformations $\tilde{A} := \beta^{1/2}(A - Bf_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$ and $\tilde{R} := (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})$. The first-order approximation in Theorem ?? can then be written

$$u_t = \bar{u} - \left(\tilde{A}\mathcal{H}_{2(T-t)-1} + \frac{\tilde{A} \left(\tilde{A}f_{\bar{u}\bar{u}}\tilde{R}^{-1} \right)^{2(T-t-1)}}{\mathcal{F}_{2(T-t)-1}\mathcal{F}_{2(T-t)}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}} \right) (x_t - \bar{x}).$$

Proof See Appendix ??. □

Corollary 3.1 If $\tilde{A}^2 = 1$, the first-order approximation of the control function simplifies to

$$u_t = \bar{u} - \left(\tilde{A}\mathcal{H}_{2(T-t)} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}} \right) (x_t - \bar{x}).$$

Proof See Appendix ??. □

Example 3.1 (The Brock-Mirman Economic Growth Model) Consider the standard textbook economic growth model often referred to as the Brock-Mirman model ²

$$\min_{\{u_t\}_{t=0}^{T-1}} - \sum_{t=0}^{T-1} \beta^t \ln(\gamma x_t^\alpha - u_t) \quad \text{s.t.} \quad x_{t+1} = u_t, \quad x_T = \bar{x}, \quad x_0 > 0. \quad (13)$$

The steady-state of this model is given by $\bar{x} = \bar{u} = (\alpha\beta\gamma)^{1/(1-\alpha)}$.³ Simplifying the example, we normalise the steady-state to unity ($\bar{x} = \bar{u} = 1$) by imposing $\beta = 1$ and $\gamma = \alpha^{-1}$. From the transition equation, $x_{t+1} = u_t$, it follows that $A = 0$ and $B = 1$. In order to make the example particularly neat we let $\alpha = 1 - \phi^{-1}$ where ϕ is the golden ratio. It then follows from the criterion function f that⁴

$$f_{\bar{x}\bar{x}} = 2(1 - \phi^{-1}), \quad f_{\bar{u}\bar{u}} = 1 - \phi^{-1}, \quad f_{\bar{x}\bar{u}} = -(1 - \phi^{-1}).$$

From these second derivatives it follows that $f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} = -1$, $\tilde{A} = 1$ and $\tilde{R} := f_{\bar{x}\bar{x}} - f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}}^2 = 1 - \phi^{-1}$. Since $\tilde{A} = 1$, we can apply Corollary ?? which yields the first-order approximation of the control function

$$u_t = 1 - (\mathcal{H}_{2(T-t)} - 1)(x_t - 1). \quad (14)$$

With the above choice of parameter values the sequence \mathcal{H} is in this example given by the original set of Fibonacci ratios H ; see (??).⁵ In Table ?? the optimal solution is compared with the control given by equation (??). At the initial time period, the discrepancy between the optimal control and the first-order approximation is 0.6 %.

	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
x_t^*	0.8000	0.9183	0.9681	0.9879	0.9960	1.0000
u_t^*	0.9183	0.9681	0.9879	0.9960	1.0000	
u_t	0.9236	0.9689	0.9880	0.9960	1.0000	
$H_{2(5-t)}$	0.6182	0.6190	0.6250	0.6667	1.0000	

Table 1: Comparing the optimal control with the first-order approximation. The first and second row provide the optimal solution to the Brock-Mirman model. In the third row the Fibonacci based control is presented as given by equation (??). The sequence in the fourth row is every second element of the original set of Fibonacci ratios (??) given in reverse.

²See Appendix ?? for some narrative details on this model.

³See Appendix ??.

⁴See Appendix ??.

⁵See Appendix ??.

4 An Explicit Solution of the Control Function in Theorem ??

In order to find an explicit solution of the control function in Theorem ??, we observe that the undetermined expressions in the control function consist of even and odd indexed generalised Fibonacci numbers only, i.e.; the sequence

$$\mathcal{H}_{2(T-t)-1} = \mathcal{F}_{2(T-t-1)} / \mathcal{F}_{2(T-t)-1},$$

has even-indexed Fibonacci numbers in the numerator and odd-indexed numbers in the denominator. The problem of finding an explicit solution of the control function is thus reduced to finding an explicit solution of the odd and even indexed Fibonacci sequence. With this goal in mind, we note that the generalised Fibonacci sequence can be written

$$\begin{aligned} \mathcal{F}_{n+2} &= a\mathcal{F}_{n+1} + b_{n+2}\mathcal{F}_n \\ &= a(a\mathcal{F}_n + b_{n+1}\mathcal{F}_{n-1}) + b_{n+2}\mathcal{F}_n \\ &= (a^2 + b_{n+2} + b_{n+1})\mathcal{F}_n - b_n b_{n+1}\mathcal{F}_{n-2}. \end{aligned}$$

Using the particular coefficient values $a = \tilde{B}$ and $b_{n+2} = f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}\tilde{A}^2$, when n is even, and $b_{n+2} = f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}$, when n is odd, yields

$$\mathcal{F}_{n+2} = (\tilde{B}^2 + f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}(\tilde{A}^2 + 1))\mathcal{F}_n - f_{\tilde{u}\tilde{u}}^2\tilde{R}^{-2}\tilde{A}^2\mathcal{F}_{n-2}. \quad (15)$$

Even though the Fibonacci sequence under consideration has time varying coefficients (b_{n+2}), the sequence describing every second generalised Fibonacci number (??) has constant coefficients, see also Byström et al. ?. Since a second-order difference equation with constant coefficients can be written in the form of (??), the solution to (??) is well known. Given the auxiliary parameters

$$c_1 := \sqrt{\tilde{B}^2 + f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}(1 - \tilde{A}^2)}, \quad c_2 := f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}\tilde{A}, \quad r_{1,2} := (c_1 \pm \sqrt{c_1^2 + 4c_2})/2, \quad (16)$$

the explicit expressions for the Fibonacci sequences entering the control function are given by⁶

$$\mathcal{F}_{2(T-t-1)} = \tilde{B} \frac{r_1^{2(T-t-1)} - r_2^{2(T-t-1)}}{r_1^2 - r_2^2}, \quad (17)$$

$$\mathcal{F}_{2(T-t)} = \tilde{B} \frac{r_1^{2(T-t)} - r_2^{2(T-t)}}{r_1^2 - r_2^2}, \quad (18)$$

$$\mathcal{F}_{2(T-t)-1} = \frac{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}}{r_1^2 - r_2^2}. \quad (19)$$

Inserting these expressions into Theorem ?? and Corollary ?? yields the following results:

Corollary 4.1 *The explicit solution of the control function given in Theorem ?? is given by*

$$u_t = \bar{u} - \left(\tilde{A}\tilde{B} \frac{r_1^{2(T-t-1)} - r_2^{2(T-t-1)}}{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}} + \tilde{A}\tilde{B}^{-1} \frac{(r_1^2 - r_2^2)^2 \left(\tilde{A}f_{\bar{u}\bar{u}}\tilde{R}^{-1} \right)^{2(T-t-1)}}{\left((r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1} \right) \left(r_1^{2(T-t)} - r_2^{2(T-t)} \right)} + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} \right) (x_t - \bar{x}).$$

Corollary 4.2 *The explicit solution of the control function given in Corollary ?? is given by*

$$u_t = \bar{u} - \left(\tilde{A}\tilde{B}^{-1} \frac{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}}{r_1^{2(T-t)} - r_2^{2(T-t)}} + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} \right) (x_t - \bar{x}).$$

Example 4.1 (The Brock-Mirman Economic Growth Model Continued) The analytical solution describing the first-order approximation of the control function in the Brock-Mirman model is given directly from Corollary ?. Since from (??) we have $c_1 = c_2 = 1$, the roots of the characteristic equation are given by

$$r_{1,2} = \frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

In terms of the golden ratio, we can write the roots as $r_1 = \phi$ and $r_2 = 1 - \phi = -\phi^{-1}$. By inserting the relationships $\tilde{A} = \tilde{B} = -f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} = \bar{x} = \bar{u} = 1$ into Corollary ?? yields the explicit expression

$$u_t = 1 - \left(\frac{\phi^{2(T-t)-1} + \phi^{1-2(T-t)}}{\phi^{2(T-t)} + \phi^{-2(T-t)}} - 1 \right) (x_t - 1).$$

⁶See Appendix ??.

5 Conclusion

In this article, we have shown how to use a generalised Fibonacci sequence for solving finite horizon dynamic optimisation problems. The solution method proposed has revealed important properties of the optimal control problem. In particular, we have shown how the first-order approximation of the optimal control function can be written in terms of these generalised Fibonacci numbers. Further, by developing the explicit solution of the generalised Fibonacci sequence, we could obtain a non-recursive solution of the first-order approximation. The procedure has been illustrated with the Brock-Mirman economic model. On a general level, we have thus bridged the area of mathematical number theory with that of optimal control.

6 Appendix

6.1 Proof: Theorem ??

We assume that standard regularity conditions of the optimal control problem hold, i.e., the criterion function f is sufficiently smooth and convex and policies, that are feasible, lie within a compact and convex set. More specific, we will from now on assume that f is twice differentiable and that the Hessian of f is positive definite. With these premises, we derive the first-order approximation of the control function by applying the perturbation control technique, as outlined in e.g., Section 4.6 in Lewis et al. ?.

The Lagrangian (\mathcal{L}) of the optimal control problem becomes

$$\mathcal{L} := \mu_{T+1}(x_T - \bar{x}) + \sum_{t=0}^{T-1} (\beta^t f(x_t, u_t) + \lambda_{t+1}(Ax_t + Bu_t - x_{t+1})), \quad (20)$$

where μ_{T+1} and λ_{t+1} represent Lagrangian multipliers. A necessary condition for optimality is that the first variation of the Lagrangian is zero. In particular, the first variation of the Lagrangian evaluated at the steady-state is zero. An optimal control minimising the Lagrangian (??) can thus be approximated by an incremental control minimising the second variation

$$\begin{aligned} d^2\mathcal{L} = d\mu_{T+1}dx_T + \frac{1}{2} \sum_{t=0}^{T-1} (\beta^t (dx_t f_{\bar{x}\bar{x}} dx_t + du_t f_{\bar{u}\bar{u}} du_t + 2du_t f_{\bar{x}\bar{u}} dx_t) \\ + 2d\lambda_{t+1}(Adx_t + Bdu_t - dx_{t+1})), \end{aligned}$$

where increments are made around the steady-state, i.e., $du_t := u_t - \bar{u}$ and $dx_t := x_t - \bar{x}$, and where e.g., the second partial derivative of f with respect to x_t , evaluated at the steady-state, is denoted by $f_{\bar{x}\bar{x}}$. This latter problem is recognised as the Lagrangian of the auxiliary discounted linear quadratic problem (DLQP)

$$\begin{aligned} \text{(DLQP)} \quad \min_{\{du_t\}_{t=0}^{T-1}} \frac{1}{2} \sum_{t=0}^{T-1} \beta^t (dx_t f_{\bar{x}\bar{x}} dx_t + du_t f_{\bar{u}\bar{u}} du_t + 2du_t f_{\bar{x}\bar{u}} dx_t) \\ \text{s.t. } dx_{t+1} = Adx_t + Bdu_t, \quad dx_T = 0, \end{aligned} \quad (21)$$

where $d\lambda_{t+1}$ and $d\mu_{T+1}$ represent the multipliers associated with the constraints (??). In order to simplify notation, we note the following identity (assuming $f_{\bar{u}\bar{u}}^{-1}$ exists)

$$\begin{aligned} dx_t f_{\bar{x}\bar{x}} dx_t + du_t f_{\bar{u}\bar{u}} du_t + 2du_t f_{\bar{x}\bar{u}} dx_t \\ = dx_t (f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}} f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}}) dx_t + (du_t + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t) f_{\bar{u}\bar{u}} (du_t + f_{\bar{u}\bar{u}}^{-1} f_{\bar{x}\bar{u}} dx_t). \end{aligned}$$

Defining $d\tilde{u}_t := (du_t + f_{\tilde{u}\tilde{u}}^{-1} f_{\tilde{x}\tilde{u}} dx_t)$ and $\tilde{R} := (f_{\tilde{x}\tilde{x}} - f_{\tilde{x}\tilde{u}} f_{\tilde{u}\tilde{u}}^{-1} f_{\tilde{x}\tilde{u}})$, the objective function in the problem (DLQP) is equivalent to

$$\frac{1}{2} \sum_{t=0}^{T-1} \beta^t \left(dx_t \tilde{R} dx_t + d\tilde{u}_t f_{\tilde{u}\tilde{u}} d\tilde{u}_t \right). \quad (22)$$

The constraint can be altered correspondingly. Inserting $du_t = d\tilde{u}_t - f_{\tilde{u}\tilde{u}}^{-1} f_{\tilde{x}\tilde{u}} dx_t$ into (??) gives

$$dx_{t+1} = (A - B f_{\tilde{u}\tilde{u}}^{-1} f_{\tilde{x}\tilde{u}}) dx_t + B d\tilde{u}_t. \quad (23)$$

In order to convert the problem to one without discounting, we define the variables $\tilde{x}_t := \beta^{t/2} dx_t$ and $\tilde{u}_t := \beta^{t/2} d\tilde{u}_t$. Substituting these newly defined variables into (??) and (??) yields the linear quadratic problem

$$\begin{aligned} \text{(LQP)} \quad & \min_{\{\tilde{u}_t\}_{t=0}^{T-1}} \frac{1}{2} \sum_{t=0}^{T-1} \left(\tilde{x}_t \tilde{R} \tilde{x}_t + \tilde{u}_t f_{\tilde{u}\tilde{u}} \tilde{u}_t \right) \\ \text{s.t.} \quad & \tilde{x}_{t+1} = \tilde{A} \tilde{x}_t + \tilde{B} \tilde{u}_t, \quad \tilde{x}_T = 0, \end{aligned} \quad (24)$$

where $\tilde{A} := \beta^{1/2}(A - B f_{\tilde{u}\tilde{u}}^{-1} f_{\tilde{x}\tilde{u}})$ and $\tilde{B} := \beta^{1/2} B$. Variables with a tilde are in the problem (LQP) thus transformed from the problem (DLQP). As a result, the problem of finding the optimal plan that minimises the problem (LQP) is equivalent to finding the optimal plan which minimises the problem (DLQP) using the appropriate transformations. The problem (LQP) is well known and its solution is given by⁷

$$\tilde{u}_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}}) \tilde{x}_t. \quad (25)$$

This control function describes the optimal control to the linear quadratic problem as a linear function of the state variable. The time varying coefficient in front of the state variable consists of two parts. The first part ($L_t^{\mathbf{a}}$) is the feedback equation of a linear quadratic problem when there is no restriction on the final state, i.e., \tilde{x}_T is free to vary. It is determined by equations (??) and (??). The last part of the feedback coefficient ($L_t^{\mathbf{b}}$) represents the linear part which ensures that the control function will drive the state to zero at the final time period and is determined by equations (??), (??) and (??).

We have now developed a linkage between the first-order approximation of the control function and a linear quadratic problem via a set of transformations. Having found a recursive solution of the linear quadratic problem we can back out the first-order approximation of the general problem by applying the set of transformations in reverse.

⁷See Appendix 6.3 in von Brasch et al. ?. For a textbook derivation see Section 4.5 in Lewis et al. ?.

The optimal solution to the problem (LQP) is given by $\tilde{u}_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}})\tilde{x}_t$. Using the definitions $\tilde{u}_t := \beta^{t/2}d\tilde{u}_t$ and $\tilde{x}_t := \beta^{t/2}dx_t$ yields

$$d\tilde{u}_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}})dx_t.$$

Further, substituting $d\tilde{u}_t := (du_t + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}dx_t)$ yields the optimal control of the problem (DLQP)

$$du_t = -(L_t^{\mathbf{a}} - L_t^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})dx_t.$$

Since increments are made around the steady-state, $du_t := u_t - \bar{u}$ and $dx_t := x_t - \bar{x}$, the first-order approximated control function of the optimal control problem can be expressed by

$$u_t = \bar{u} - (L_t^{\mathbf{a}} - L_t^{\mathbf{b}} + f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})(x_t - \bar{x}), \quad (26)$$

where $L_t^{\mathbf{a}}$ and $L_t^{\mathbf{b}}$ are given by (??) - (??). This linearised control function ensures that the state reach the steady-state in the final period, i.e., restriction (??) holds also for this control function.⁸ \square

6.2 Proof: Theorem ??

The first-order approximation (??) consists of two sequences $L_t^{\mathbf{a}}$ and $L_t^{\mathbf{b}}$. We show the linkage between the generalised Fibonacci sequence and these sequences separately.

6.2.1 Fibonacci and Optimal Control: $L_t^{\mathbf{a}}$

First, we note that the ratio of Fibonacci numbers, $\mathcal{H}_n = \mathcal{F}_{n-1}/\mathcal{F}_n$, can also be generated by

$$\mathcal{H}_{n+2} = \frac{a + b_{n+1}\mathcal{H}_n}{a^2 + b_{n+2} + ab_{n+1}\mathcal{H}_n}, \quad (27)$$

with initial value $\mathcal{H}_1 = 0$. Further, combining (??) with (??) we can write $S_{t+1} = f_{\bar{u}\bar{u}}\tilde{A}\tilde{B}^{-1}L_{t+1}^{\mathbf{a}} + \tilde{R}$, which when inserted into (??) yields

$$\tilde{A}^{-1}L_t^{\mathbf{a}} = \frac{\tilde{B} + f_{\bar{u}\bar{u}}\tilde{R}^{-1}\tilde{A}^2(\tilde{A}^{-1}L_{t+1}^{\mathbf{a}})}{\tilde{B}^2 + f_{\bar{u}\bar{u}}\tilde{R}^{-1} + \tilde{B}f_{\bar{u}\bar{u}}\tilde{R}^{-1}\tilde{A}^2(\tilde{A}^{-1}L_{t+1}^{\mathbf{a}})}. \quad (28)$$

Comparing (??) with (??) we note that using the particular values $a = \tilde{B}$ and $b_{n+2} = f_{\bar{u}\bar{u}}\tilde{R}^{-1}\tilde{A}^2$, when n is even, and $b_{n+2} = f_{\bar{u}\bar{u}}\tilde{R}^{-1}$, when n is odd, makes (??) identical with the sequence of the transformed

⁸See Appendix 6.4 in von Brasch et al. ?.

feedback (??) with appropriate change of index. The sequence $\tilde{A}^{-1}L_t^{\mathbf{a}}$ runs backward from an initial value at time $t = T - 1$. If we make the index change $n = 2(T - t) - 1$, the sequence $\mathcal{H}_n = \mathcal{H}_{2(T-t)-1}$ begins at the initial value $\mathcal{H}_1 = 0$. Since from (??) the initial value of the feedback equation is zero, and consequently $\tilde{A}^{-1}L_{T-1} = 0$, we have derived the following relationship

$$L_t^{\mathbf{a}} = \tilde{A} \mathcal{H}_{2(T-t)-1}, \quad 0 \leq t \leq T - 1. \quad (29)$$

6.2.2 Fibonacci and Optimal Control: $L_t^{\mathbf{b}}$

In order to derive the relationship between the second part of the control function ($L_t^{\mathbf{b}}$) and the generalised Fibonacci sequence, we note that the inverse of (??) can be written

$$\mathcal{H}_{n+2}^{-1} = \frac{(a^2 + b_{n+2})\mathcal{H}_n^{-1} + ab_{n+1}}{a\mathcal{H}_n^{-1} + b_{n+1}}. \quad (30)$$

Multiplying the Riccati equation (??) by $(\tilde{B}\tilde{R}^{-1})$ yields

$$(\tilde{B}\tilde{R}^{-1}S_t) = \frac{(\tilde{B}^2 + f_{\bar{u}\bar{u}}R^{-1}\tilde{A}^2)(\tilde{B}\tilde{R}^{-1}S_{t+1}) + B\tilde{R}^{-1}f_{\bar{u}\bar{u}}}{\tilde{B}(\tilde{B}\tilde{R}^{-1}S_{t+1}) + \tilde{R}^{-1}f_{\bar{u}\bar{u}}}. \quad (31)$$

We note that the same choice of coefficients as in Section ?? makes the sequence (??) identical to the sequence (??), i.e., $a = \tilde{B}$ and $b_{n+2} = f_{\bar{u}\bar{u}}\tilde{R}^{-1}\tilde{A}^2$, when n is even, and $b_{n+2} = f_{\bar{u}\bar{u}}\tilde{R}^{-1}$, when n is odd. The sequence $(\tilde{B}\tilde{R}^{-1}S_t)$ runs backward from time ($t = T$) with an initial condition which follows from the Riccati equation $(\tilde{B}\tilde{R}^{-1}S_T) = 0$. Since $\mathcal{F}_0/\mathcal{F}_{-1} = 0$, we define $\mathcal{H}_0^{-1} := 0$, even though \mathcal{H}_0 is undefined. This gives the following relationship between the solution of the Riccati equation and the ratio of Fibonacci sequences, for $0 \leq t \leq T$,

$$S_t = \tilde{R}\tilde{B}^{-1}\mathcal{H}_{2(T-t)}^{-1}. \quad (32)$$

Further, we note that from (??) and (??)

$$\begin{aligned} W_{T-1} &= \tilde{A} \left(1 - \tilde{B}\mathcal{H}_1\right) W_T = \tilde{A}, \\ P_{T-1} &= P_T - \frac{W_T^2 \tilde{B}^2}{f_{\bar{u}\bar{u}} + \tilde{B}^2 S_T} = -\frac{\tilde{B}^2}{f_{\bar{u}\bar{u}}}, \end{aligned}$$

hence the initial condition L_{T-1}^b is, from (??)

$$L_{T-1}^b = \left(f_{\bar{u}\bar{u}} + \tilde{B}^2 S_T \right)^{-1} \tilde{B} W_T P_{T-1}^{-1} W_{T-1} = -\frac{\tilde{A}\tilde{B}}{f_{\bar{u}\bar{u}} \frac{\tilde{B}^2}{f_{\bar{u}\bar{u}}}} = -\frac{\tilde{A}}{\tilde{B}}.$$

For $k = 0, 1, 2, 3, \dots$, we can rewrite the sequence of generalised Fibonacci numbers (\mathcal{F}_n)

$$\mathcal{F}_{2k+2} = \tilde{B}\mathcal{F}_{2k+1} + \frac{\tilde{A}^2 f_{\bar{u}\bar{u}}}{\tilde{R}} \mathcal{F}_{2k}, \quad \mathcal{F}_0 = 0, \quad \mathcal{F}_1 = 1, \quad (33)$$

$$\mathcal{F}_{2k+1} = \tilde{B}\mathcal{F}_{2k} + \frac{f_{\bar{u}\bar{u}}}{\tilde{R}} \mathcal{F}_{2k-1}, \quad \mathcal{F}_0 = 0, \quad \mathcal{F}_{-1} = \frac{\tilde{R}}{f_{\bar{u}\bar{u}}}. \quad (34)$$

With these premises, we want to show that also the second feedback coefficient can be explicitly expressed in terms of generalised Fibonacci numbers; more specific, we have that

$$W_{T-k} = \left(\frac{\tilde{A}f_{\bar{u}\bar{u}}}{\tilde{R}} \right)^{k-1} \frac{\tilde{A}}{\mathcal{F}_{2k-1}}, \quad k = 0, 1, 2, \dots, T, \quad (35)$$

$$P_{T-k} = -\frac{\tilde{B}}{f_{\bar{u}\bar{u}}} \mathcal{H}_{2k}^{-1}, \quad k = 0, 1, 2, \dots, T, \quad (36)$$

$$L_{T-k}^b = -\left(\frac{\tilde{A}f_{\bar{u}\bar{u}}}{\tilde{R}} \right)^{2(k-1)} \frac{\tilde{A}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}}, \quad k = 1, 2, \dots, T. \quad (37)$$

To this end, we use the principle of induction. Having in mind that

$$\begin{aligned} \mathcal{F}_{-1} &= \frac{\tilde{R}}{f_{\bar{u}\bar{u}}}, \\ \mathcal{H}_0^{-1} &= 0, \end{aligned}$$

we see that the initial conditions are satisfied since

$$\begin{aligned} W_T &= \left(\frac{\tilde{A}f_{\bar{u}\bar{u}}}{\tilde{R}} \right)^{-1} \frac{\tilde{A}}{\frac{\tilde{R}}{f_{\bar{u}\bar{u}}}} = 1, \\ P_T &= -\frac{\tilde{B}}{f_{\bar{u}\bar{u}}} \mathcal{H}_0^{-1} = 0, \\ L_{T-1}^b &= -\left(\frac{\tilde{A}f_{\bar{u}\bar{u}}}{\tilde{R}} \right)^0 \frac{\tilde{A}}{\mathcal{F}_1\mathcal{F}_2} = -\frac{\tilde{A}}{\tilde{B}}. \end{aligned} \quad (38)$$

In the following, we show that if the expressions (?? - ??) are true for $k = p$, then they are also true for $k = p + 1$. Indeed, equation (??) with (??) yields that

$$\begin{aligned} W_{T-(p+1)} &= \tilde{A} \left(1 - \tilde{B} \mathcal{H}_{2p+1}\right) W_{T-p} = \tilde{A}^2 \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{p-1} \left(1 - \tilde{B} \frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p+1}}\right) \frac{1}{\mathcal{F}_{2p-1}} \\ &= \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^p \frac{\tilde{R}}{f_{\tilde{u}\tilde{u}}} \frac{\mathcal{F}_{2p+1} - \tilde{B} \mathcal{F}_{2p}}{\mathcal{F}_{2p-1}} \frac{\tilde{A}}{\mathcal{F}_{2p+1}} = \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^p \frac{\tilde{A}}{\mathcal{F}_{2(p+1)-1}}, \end{aligned}$$

where the last equality follows from (??). Moreover, equation (??) with (??) yields that

$$\begin{aligned} P_{T-(p+1)} &= P_{T-p} - \tilde{B}^2 W_{T-p}^2 \left(f_{\tilde{u}\tilde{u}} + \tilde{R} \tilde{B} \mathcal{H}_{2p}^{-1}\right)^{-1} \\ &= -\frac{\tilde{B}}{f_{\tilde{u}\tilde{u}}} \mathcal{H}_{2p}^{-1} - \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2(p-1)} \frac{\tilde{B}^2 \tilde{A}^2}{\mathcal{F}_{2p-1}^2 \left(f_{\tilde{u}\tilde{u}} + \tilde{R} \tilde{B} \frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}}\right)} \\ &= -\frac{\tilde{B}}{f_{\tilde{u}\tilde{u}}} \left(\frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}} + \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2p-1} \frac{\tilde{A} \tilde{B}}{\mathcal{F}_{2p-1} \mathcal{F}_{2p+1}}\right) \\ &= -\frac{\tilde{B}}{f_{\tilde{u}\tilde{u}}} \left(\frac{\mathcal{F}_{2p}}{\mathcal{F}_{2p-1}} + \frac{\mathcal{F}_{2p+2} \mathcal{F}_{2p-1} - \mathcal{F}_{2p} \mathcal{F}_{2p+1}}{\mathcal{F}_{2p-1} \mathcal{F}_{2p+1}}\right) \\ &= -\frac{\tilde{B}}{f_{\tilde{u}\tilde{u}}} \frac{\mathcal{F}_{2(p+1)}}{\mathcal{F}_{2p+1}} = -\frac{\tilde{B}}{f_{\tilde{u}\tilde{u}}} \mathcal{H}_{2(p+1)}^{-1}, \end{aligned}$$

where we have used the relation (??), corresponding to d'Ocagne's identity for regular Fibonacci numbers. Hence expressions (??) and (??) follow by the induction principle. Finally, expression (??) together with (??) for $k = 2, 3, \dots, T$, gives

$$\begin{aligned} L_{T-k}^b &= \left(f_{\tilde{u}\tilde{u}} + \tilde{R} \tilde{B} \mathcal{H}_{2(k-1)}^{-1}\right)^{-1} \tilde{B} W_{T-(k-1)} P_{T-k}^{-1} W_{T-k} \\ &= \frac{\tilde{B} \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{k-2} \tilde{A} \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{k-1} \tilde{A}}{\frac{\tilde{B}}{f_{\tilde{u}\tilde{u}}} \mathcal{H}_{2k}^{-1} \left(f_{\tilde{u}\tilde{u}} + \tilde{R} \tilde{B} \frac{\mathcal{F}_{2(k-1)}}{\mathcal{F}_{2k-3}}\right) \mathcal{F}_{2k-3} \mathcal{F}_{2k-1}} \\ &= \frac{f_{\tilde{u}\tilde{u}} \tilde{A}^2 \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2k-3}}{\mathcal{F}_{2k} \left(f_{\tilde{u}\tilde{u}} \mathcal{F}_{2k-3} + \tilde{R} \tilde{B} \mathcal{F}_{2(k-1)}\right)} = \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2(k-1)} \frac{\tilde{A}}{\mathcal{F}_{2k} \mathcal{F}_{2k-1}}, \end{aligned}$$

where the last equality follows from (??).

In proving the explicit expression for P_{T-k} , we used the identity

$$\mathcal{F}_{2k+2} \mathcal{F}_{2k-1} - \mathcal{F}_{2k} \mathcal{F}_{2k+1} = \tilde{A} \tilde{B} \left(\frac{\tilde{A} f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2k-1}, \quad k = 0, 1, 2, \dots \quad (39)$$

This identity is also proved by using induction. First, we note that the initial condition is satisfied since

$$\mathcal{F}_2\mathcal{F}_{-1} - \mathcal{F}_0\mathcal{F}_1 = \tilde{B}\frac{\tilde{R}}{f_{\tilde{u}\tilde{u}}} - 0 \cdot 1 = \tilde{A}\tilde{B}\left(\frac{\tilde{A}f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{-1}.$$

Now, let us assume that the identity is true for $k = p$, that is,

$$\mathcal{F}_{2p+2}\mathcal{F}_{2p-1} - \mathcal{F}_{2p}\mathcal{F}_{2p+1} = \tilde{A}\tilde{B}\left(\frac{\tilde{A}f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2p-1}.$$

The proof is complete if we can show that it also holds for $k = p + 1$. Indeed,

$$\begin{aligned} & \mathcal{F}_{2p+4}\mathcal{F}_{2p+1} - \mathcal{F}_{2p+2}\mathcal{F}_{2p+3} \\ &= \left(\tilde{B}\mathcal{F}_{2p+3} + \frac{\tilde{A}^2 f_{\tilde{u}\tilde{u}}}{\tilde{R}}\mathcal{F}_{2p+2}\right)\mathcal{F}_{2p+1} - \left(\tilde{B}\mathcal{F}_{2p+1} + \frac{\tilde{A}^2 f_{\tilde{u}\tilde{u}}}{\tilde{R}}\mathcal{F}_{2p}\right)\mathcal{F}_{2p+3} \\ &= \frac{\tilde{A}^2 f_{\tilde{u}\tilde{u}}}{\tilde{R}}(\mathcal{F}_{2p+2}\mathcal{F}_{2p+1} - \mathcal{F}_{2p}\mathcal{F}_{2p+3}) \\ &= \frac{\tilde{A}^2 f_{\tilde{u}\tilde{u}}}{\tilde{R}}\left(\mathcal{F}_{2p+2}\left(\tilde{B}\mathcal{F}_{2p} + \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\mathcal{F}_{2p-1}\right) - \mathcal{F}_{2p}\left(\tilde{B}\mathcal{F}_{2p+2} + \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\mathcal{F}_{2p+1}\right)\right) \\ &= \frac{\tilde{A}^2 f_{\tilde{u}\tilde{u}}^2}{\tilde{R}^2}(\mathcal{F}_{2p+2}\mathcal{F}_{2p-1} - \mathcal{F}_{2p}\mathcal{F}_{2p+1}) = \tilde{A}\tilde{B}\left(\frac{\tilde{A}f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2p+1}, \end{aligned}$$

where the last equality follows from the induction assumption. Changing index, we have thus shown how the Fibonacci sequence enters the second feedback term

$$L_t^{\mathbf{b}} = -\frac{\tilde{A}\left(\tilde{A}f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}\right)^{2(T-t-1)}}{\mathcal{F}_{2(T-t)-1}\mathcal{F}_{2(T-t)}}, \quad 0 \leq t \leq T-1. \quad (40)$$

□

6.3 Proof: Corollary ??

Since

$$\begin{aligned} L_{T-k}^{\mathbf{a}} &= \tilde{A}\mathcal{H}_{2k-1}, \\ L_{T-k}^{\mathbf{b}} &= -\left(\frac{\tilde{A}f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2(k-1)} \frac{\tilde{A}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}}, \end{aligned}$$

the control function

$$\tilde{u}_{T-k} = -(L_{T-k}^{\mathbf{a}} - L_{T-k}^{\mathbf{b}})\tilde{x}_{T-k},$$

can be simplified when $\tilde{A}^2 = 1$. First, note that

$$\begin{aligned} L_{T-k}^{\mathbf{a}} - L_{T-k}^{\mathbf{b}} &= \tilde{A} \left(\mathcal{H}_{2k-1} + \frac{\left(\frac{\tilde{A}f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2(k-1)}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}} \right) \\ &= \tilde{A} \frac{\mathcal{F}_{2k-2}\mathcal{F}_{2k} + \left(\tilde{A}^2\right)^{k-1} \left(\frac{\tilde{A}f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2(k-1)}}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}}. \end{aligned}$$

If we let $\tilde{A}^2 = 1$, we then get that

$$L_{T-k}^{\mathbf{a}} - L_{T-k}^{\mathbf{b}} = \tilde{A} \frac{\mathcal{F}_{2k-1}^2}{\mathcal{F}_{2k-1}\mathcal{F}_{2k}} = \tilde{A}\mathcal{H}_{2k},$$

by noting that

$$\mathcal{F}_{2k-1}^2 - \mathcal{F}_{2k-2}\mathcal{F}_{2k} = \left(\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{2(k-1)}, \quad k = 1, 2, 3, \dots,$$

which follows from setting $n = 2k - 1$ in the identity

$$\mathcal{F}_n^2 - \mathcal{F}_{n-1}\mathcal{F}_{n+1} = \left(-\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{n-1}, \quad n = 1, 2, 3, \dots \quad (41)$$

This identity is proved by using induction. First, we note that the initial condition is satisfied since

$$\mathcal{F}_1^2 - \mathcal{F}_0\mathcal{F}_2 = 1 = \left(-\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^0.$$

Now, let us assume that the identity is true for $n = p$, that is,

$$\mathcal{F}_p^2 - \mathcal{F}_{p-1}\mathcal{F}_{p+1} = \left(-\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{p-1}.$$

The proof is complete if we can show that it then also holds for $n = p + 1$. Indeed, by using $\tilde{A}^2 = 1$, we get

$$\begin{aligned} \mathcal{F}_{p+1}^2 - \mathcal{F}_p\mathcal{F}_{p+2} &= \left(\tilde{B}\mathcal{F}_p + \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\mathcal{F}_{p-1}\right)\mathcal{F}_{p+1} - \left(\tilde{B}\mathcal{F}_{p+1} + \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\mathcal{F}_p\right)\mathcal{F}_p \\ &= \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}(\mathcal{F}_{p+1}\mathcal{F}_{p-1} - \mathcal{F}_p^2) = \left(-\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)\left(-\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^{p-1} = \left(-\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}}\right)^p, \end{aligned}$$

where the penultimate equality follows from the induction assumption.

Remark 6.1 Identity (??) is a generalisation of Cassini's identity

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}, \quad n = 1, 2, 3, \dots,$$

for regular Fibonacci numbers.

Hence in this special case, we have that

$$\tilde{u}_{T-k} = - (L_{T-k}^{\mathbf{a}} - L_{T-k}^{\mathbf{b}}) \tilde{x}_{T-k} = -\tilde{A}\mathcal{H}_{2k}\tilde{x}_{T-k},$$

or

$$\tilde{u}_t = - (L_t^{\mathbf{a}} - L_t^{\mathbf{b}}) \tilde{x}_t = -\tilde{A}\mathcal{H}_{2(T-t)}\tilde{x}_t.$$

□

6.4 Example: Narrative Details on the Brock-Mirman Model

The Brock-Mirman model considers a representative household maximising utility subject to economic constraints.⁹ In particular, it considers an economy where the total amount of goods (y_t) is produced using capital (x_t) as input in the production process, i.e.,

$$y_t = \gamma x_t^\alpha, \quad (42)$$

where $\gamma > 0$ and $0 < \alpha < 1$. In a closed economy, what is produced in a given year must either be consumed (c_t) or invested (u_t) as given by the national accounts identity

$$y_t = c_t + u_t. \quad (43)$$

Further, if we make the simplifying assumption that capital fully depreciates, the consecutive level of capital will equal current investments, i.e.,

$$x_{t+1} = u_t. \quad (44)$$

Given an initial level of capital (x_0), the objective of the representative household is to maximise a discounted ($0 < \beta < 1$) sum of utilities

$$\sum_{t=0}^{T-1} \beta^t \ln(c_t), \quad (45)$$

⁹A representative household refers to an economy where the demand side can be represented as if there were a single household making the aggregate consumption and saving decisions.

subject to the three economic constraints (??) - (??) and subject to capital reaching the steady-state value in the final time period

$$x_T = \bar{x}. \tag{46}$$

The form of the Brock-Mirman model as given in the main text follows by inserting both the production function (??) and the national accounts identity (??) into the objective function (??). More details on the Brock-Mirman model can be found in Section 3.1.2 in Ljungqvist and Sargent ?.

6.5 Example: Deriving the Steady-State

In this section, we derive the steady-state of the Brock-Mirman model. Define the Hamiltonian

$$H := -\beta^t \ln(\gamma x_t^\alpha - u_t) + \lambda_{t+1} u_t,$$

where λ_{t+1} is the multiplier. The first-order conditions are¹⁰

$$\begin{aligned} H_{u_t} = 0 &\quad \Rightarrow \quad -\beta^t (\gamma x_t^\alpha - u_t)^{-1} = \lambda_{t+1}, \\ \lambda_t = H_{x_t} &\quad \Rightarrow \quad \lambda_t = -\beta^t \alpha \gamma x_t^{\alpha-1} (\gamma x_t^\alpha - u_t)^{-1}. \end{aligned}$$

Combining these first-order conditions, letting $c_t = \gamma x_t^\alpha - u_t$, yields the Euler-Lagrange equation

$$c_{t+1} = c_t \beta \alpha \gamma x_{t+1}^{\alpha-1}.$$

At the steady-state, both the control and the state remains unchanged, $\bar{c} = c_t = c_{t+1}$ and $\bar{x} = x_t = x_{t+1}$.

The Euler equation can thus be solved to yield

$$\bar{x} = (\alpha \beta \gamma)^{1/(1-\alpha)}.$$

Further, the steady-state levels of investment and consumption are given by

$$\bar{u} = \bar{x}, \quad \bar{c} = \gamma \bar{x}^\alpha - \bar{x}.$$

¹⁰See Section 12.4 in Sydsaeter et al. ?.

6.6 Example: Second Derivatives

In this section, we provide the second derivatives of the criterion function evaluated at the steady state. In particular, we have

$$\begin{aligned} f_{\bar{x}\bar{x}} &= \alpha^2 \gamma^2 \bar{c}^{(-2)} \bar{x}^{2(\alpha-1)} + \alpha(1-\alpha) \gamma \bar{c}^{(-1)} \bar{x}^{(\alpha-2)}, \\ f_{\bar{u}\bar{u}} &= \bar{c}^{(-2)}, \\ f_{\bar{x}\bar{u}} &= -\alpha \gamma \bar{c}^{(-2)} \bar{x}^{\alpha-1}. \end{aligned}$$

From Appendix ??, $\bar{c} = (1-\alpha)\alpha^{-1}$ when imposing the restrictions $\beta = 1$ and $\gamma = \alpha^{-1}$. Also inserting the normalisation $\bar{x} = 1$ and $\alpha = 1 - \phi^{-1}$ yields the results given in the main text

$$f_{\bar{x}\bar{x}} = 2(1 - \phi^{-1}), \quad f_{\bar{u}\bar{u}} = 1 - \phi^{-1}, \quad f_{\bar{x}\bar{u}} = -(1 - \phi^{-1}).$$

In order to derive $f_{\bar{x}\bar{x}}$ we have used the property

$$\alpha(1-\alpha)^{-2} = (1-\phi^{-1})\phi^2 = (\phi-1)\phi = 1.$$

where the last equality follows from applying (??).

6.7 Example: Generalised Fibonacci Sequence

In this section, we illustrate that the Fibonacci sequence entering the control function of the Brock-Mirman model is the original Fibonacci sequence. The generalised Fibonacci sequence is in this example defined by

$$\mathcal{F}_{n+2} = a\mathcal{F}_{n+1} + b_{n+2}\mathcal{F}_n, \quad \mathcal{F}_0 = 0, \quad \mathcal{F}_1 = 1,$$

with the particular coefficients $a = \beta^{1/2}B$ and $b_{n+2} = f_{\bar{u}\bar{u}}(f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})^{-1}\beta(A - Bf_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}})^2$, when n is even, and $b_{n+2} = f_{\bar{u}\bar{u}}(f_{\bar{x}\bar{x}} - f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}^2)^{-1}$, when n is odd. It follows immediately that $a = 1$. To find the expression for b_{n+2} we need the second derivatives of the criterion function. From Appendix ?? it follows that $f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}} = -1$ and $f_{\bar{x}\bar{x}} - f_{\bar{u}\bar{u}}^{-1}f_{\bar{x}\bar{u}}^2 = (1 - \phi^{-1})$. Using the parameter values $A = 0$ and $\beta = B = 1$ yields $b_{n+2} = 1$.

6.8 Section ??: Explicit Solutions of Odd and Even Indexed Fibonacci Sequences

Since (??) has constant coefficients there is an explicit solution describing this sequence. Consider the recurrence equation

$$g_{n+2} = c_1 g_{n+1} + c_2 g_n \quad (47)$$

$$\begin{aligned} &= c_1(c_1 g_n + c_2 g_{n-1}) + c_2 g_n \\ &= (c_1^2 + c_2)g_n + c_1 c_2 (c_1^{-1}(g_n - c_2 g_{n-2})) \\ &= (c_1^2 + 2c_2)g_n - c_2^2 g_{n-2}. \end{aligned} \quad (48)$$

We note that (??) describes the sequence (??) when coefficients are matched, i.e.,

$$c_1 = \sqrt{\tilde{B}^2 + f_{\tilde{u}\tilde{u}} \tilde{R}^{-1} (1 - \tilde{A})^2}, \quad c_2 = f_{\tilde{u}\tilde{u}} \tilde{R}^{-1} \tilde{A}. \quad (49)$$

The general solution to the sequence g_n is well known and depends on whether the characteristic equation $r^2 = c_1 r + c_2$ has two distinct real roots, one real double root or a pair of complex conjugate roots. Due to the assumption of a positive definite Hessian of f in the optimal control problem, it is only the real distinct roots which are relevant, i.e., $r_{1,2} = (c_1 \pm \sqrt{c_1^2 + 4c_2})/2$.¹¹ Given two initial values, the general solution is then given by

$$g_n = G r_1^n + K r_2^n, \quad (50)$$

where G and K are constants to be determined from the initial conditions. These initial conditions depend on whether we are considering the odd or even indexed Fibonacci sequence.

Let g_n^e denote the solution to equation (??) with initial values corresponding to even indexed Fibonacci numbers. For example, consider the initial value of the sequence $\mathcal{F}_{2(T-t-1)}$ when time is running backwards from $t = T - 1$, i.e., $g_0^e = \mathcal{F}_0$ and $g_2^e = \mathcal{F}_2$. Given these initial conditions, solving for the constants G and K in (??), gives the solution¹²

$$g_n^e = \tilde{B} \frac{r_1^n - r_2^n}{r_1^2 - r_2^2}.$$

In terms of the Fibonacci sequence, $g_n^e = \mathcal{F}_n$ when n is even. The explicit expressions for the even indexed

¹¹See Appendix ??.

¹²See Appendix ??.

Fibonacci sequences entering the control function are then given by

$$\mathcal{F}_{2(T-t-1)} = g_{2(T-t-1)}^e = \tilde{B} \frac{r_1^{2(T-t-1)} - r_2^{2(T-t-1)}}{r_1^2 - r_2^2}, \quad (51)$$

$$\mathcal{F}_{2(T-t)} = g_{2(T-t)}^e = \tilde{B} \frac{r_1^{2(T-t)} - r_2^{2(T-t)}}{r_1^2 - r_2^2}. \quad (52)$$

Correspondingly, we let g_n^o denote the solution to equation (??) with initial values corresponding to the initial value of the odd indexed Fibonacci numbers, i.e., $g_1^o = \mathcal{F}_1$ and $g_{-1}^o = \mathcal{F}_{-1}$. Given these initial conditions, solving for the constants G and K in (??), gives the solution¹³

$$g_n^o = \frac{(r_1 + \tilde{A}r_2)r_1^n - (r_2 + \tilde{A}r_1)r_2^n}{r_1^2 - r_2^2}.$$

Since $g_n^o = \mathcal{F}_n$ when n is odd, the explicit solution of the odd indexed Fibonacci sequence is then given by

$$\mathcal{F}_{2(T-t)-1} = \frac{(r_1 + \tilde{A}r_2)r_1^{2(T-t)-1} - (r_2 + \tilde{A}r_1)r_2^{2(T-t)-1}}{r_1^2 - r_2^2}. \quad (53)$$

6.9 Section ??: Both Roots Are Real and Distinct

This section shows that both roots of the characteristic equation corresponding to the solution of every second generalised Fibonacci sequence are real and distinct due to the assumption of a positive definite Hessian of f . The general solution to the sequence

$$g_{n+2} = c_1 g_{n+1} + c_2 g_n,$$

is well known and depends on whether the characteristic equation $r^2 = c_1 r + c_2$ has two distinct real roots, one real double root or a pair of complex conjugate roots. Due to the assumption of a positive definite Hessian of the criterion function f in the optimal control problem, we show that it is only the real and distinct roots which are relevant, i.e., $c_1^2 + 4c_2 > 0$. Indeed, given the expressions for c_1 and c_2 it follows that

$$c_1^2 + 4c_2 = \tilde{B}^2 + \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}} (1 - \tilde{A})^2 + 4 \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}} \tilde{A} = \tilde{B}^2 + \frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}} (1 + \tilde{A})^2 > 0,$$

if $\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}} > 0$. This holds since

$$\frac{f_{\tilde{u}\tilde{u}}}{\tilde{R}} = \frac{f_{\tilde{u}\tilde{u}}}{f_{\tilde{x}\tilde{x}} - f_{\tilde{x}\tilde{u}} f_{\tilde{u}\tilde{u}}^{-1} f_{\tilde{x}\tilde{u}}} = \frac{f_{\tilde{u}\tilde{u}}^2}{f_{\tilde{u}\tilde{u}} f_{\tilde{x}\tilde{x}} - f_{\tilde{x}\tilde{u}}^2} > 0,$$

¹³See Appendix ??.

which is positive from the positive definiteness of the Hessian, implying

$$\begin{vmatrix} f_{\bar{x}\bar{x}} & f_{\bar{x}\bar{u}} \\ f_{\bar{x}\bar{u}} & f_{\bar{u}\bar{u}} \end{vmatrix} = f_{\bar{u}\bar{u}}f_{\bar{x}\bar{x}} - f_{\bar{x}\bar{u}}^2 > 0 \quad \text{and} \quad f_{\bar{u}\bar{u}} > 0.$$

6.10 Section ??: The General Solution: g_n

The general solution to the difference equation

$$g_{n+2} = c_1 g_{n+1} + c_2 g_n, \tag{54}$$

when both roots of the characteristic equation $r^2 = c_1 r + c_2$ are real and distinct, is given by

$$g_n = G r_1^n + K r_2^n, \tag{55}$$

where the constants G and K are determined by initial conditions. We consider the case of even and odd indexed Fibonacci sequences separately, i.e., we find the sequences $g_n^e = \mathcal{F}_n$ when n is even and $g_n^o = \mathcal{F}_n$ when n is odd.¹⁴

6.10.1 Section ??: The Even Indexed Sequence: g_n^e

From the generalised Fibonacci sequence, $g_0^e = \mathcal{F}_0 = 0$, and

$$g_2^e = \mathcal{F}_2 = a_2 \mathcal{F}_1 + b_2 \mathcal{F}_0 = a_2 = \tilde{B},$$

which, when inserted into (??), gives the initial condition

$$g_1^e = c_1^{-1}(g_2^e - c_2 g_0^e) = c_1^{-1} g_2^e = \frac{\tilde{B}}{r_1 + r_2},$$

where we have used the property $c_1 = r_1 + r_2$. From the general solution (??) we get

$$g_0^e = 0 = G^e r_1^0 + K^e r_2^0 \Rightarrow G^e = -K^e.$$

¹⁴The superscript e emphasise that the sequence relates to the even indexed Fibonacci sequence (g^e , G^e and K^e) while the superscript o refers to the odd indexed sequence.

Inserting this result when applying the second initial condition g_1^e yields

$$g_1^e = G^e r_1 + K^e r_2 = K^e (r_2 - r_1) = \frac{\tilde{B}}{r_1 + r_2}.$$

Together, this implies

$$G^e = \frac{\tilde{B}}{r_1^2 - r_2^2}, \quad K^e = -\frac{\tilde{B}}{r_1^2 - r_2^2}.$$

Using these results in the general solution (??) gives

$$g_n^e = \tilde{B} \frac{r_1^n - r_2^n}{r_1^2 - r_2^2}.$$

Remark 6.2 Note that $r_1^2 - r_2^2 = c_1 \sqrt{c_1^2 + 4c_2} > 0$.

6.10.2 Section ??: The Odd Indexed Sequence: g_n^o

From the generalised Fibonacci sequence, $g_1^o = \mathcal{F}_1 = 1$ and $g_{-1}^o = \mathcal{F}_{-1} = \tilde{R}f_{\tilde{u}\tilde{u}}^{-1}$, which gives

$$g_0^o = (1 - \tilde{A})c_1^{-1},$$

when inserted into (??) and applying the matched coefficient $c_2 = f_{\tilde{u}\tilde{u}}\tilde{R}^{-1}\tilde{A}$. In order to determine G^o and K^o we use the initial values g_0^o and g_1^o . From the general solution (??) we get

$$g_0^o = (1 - \tilde{A})c_1^{-1} = G^o r_1^0 + K^o r_2^0,$$

or

$$G^o = (1 - \tilde{A})c_1^{-1} - K^o. \tag{56}$$

From the other initial condition, we get

$$g_1^o = G^o r_1 + K^o r_2 = 1,$$

which, when inserting (??) and using the relation $c_1 = r_1 + r_2$, yields

$$K^o = -\frac{r_2 + \tilde{A}r_1}{r_1^2 - r_2^2}.$$

Inserting this result back into (??) gives

$$G^o = \frac{r_1 + \tilde{A}r_2}{r_1^2 - r_2^2}.$$

Using these results in the general solution (??) gives

$$g_n^o = G^o r_1^n + K^o r_2^n = \frac{(r_1 + \tilde{A}r_2)r_1^n - (r_2 + \tilde{A}r_1)r_2^n}{r_1^2 - r_2^2}.$$

References

- McReynolds, S. R.: A Successive Sweep Method for Solving Optimal Programming Problems, Ph.D. Thesis, Harvard University (1966)
- Lystad, L. P.: Bruk av reguleringstekniske metoder for analyse og utvikling av økonomiske modeller (The use of control theory for analysis and development of economic models), Ph.D. Thesis, NTH, Institutt for sosialøkonomi. p. 228, Meddelse nr. 28 (1975)
- Magill, M. J. P.: A local analysis of n-sector capital accumulation under uncertainty, *Journal of Economic Theory* 15(1), 211–219 (1977)
- Magill, M. J. P.: Some new results on the local stability of the process of capital accumulation, *Journal of Economic Theory* 15(1), 174–210 (1977)
- Judd, K. L.: *Numerical Methods in Economics*, The MIT Press (1998)
- Levine, P., Pearlman, J., Pierser, R.: Linear-quadratic approximation, external habit and targeting rules, *Journal of Economic Dynamics and Control* 32(10), 3315–3349 (2008)
- Benigno, P., Woodford, M.: Linear-quadratic approximation of optimal policy problems, Discussion Papers 0809-01, Columbia University, Department of Economics. (2008)
- Benavoli, A., Chisci, L., Farina, A.: Fibonacci sequence, golden section, kalman filter and optimal control, *Signal Processing* 89(8), 1483–1488 (2009)
- Capponi, A., Farina, A., Pilotto, C.: Expressing stochastic filters via number sequences, *Signal Processing* 90(7), 2124–2132 (2010)
- Donoghue, J.: State estimation and control of the Fibonacci system, *Signal Processing* 91(5), 1190–1193 (2011)
- Byström, J., Lystad, L. P., Nyman, P.-O.: Using Generalized Fibonacci Sequences for Solving the One-Dimensional LQR Problem and its Discrete-Time Riccati Equation, *Modeling, Identification and Control* 31(1), 1–18 (2010)
- Ljungqvist, L., Sargent, T. J.: *Recursive Macroeconomic Theory*, 2nd edn, The MIT Press (2004)
- Castellanos, D.: Rapidly converging expansions with Fibonacci coefficients, *Fibonacci Quart* 24, 70–82 (1986)
- Brock, W. A., Mirman, L. J.: Optimal economic growth and uncertainty: The discounted case, *Journal of Economic Theory* 4(3), 479–513 (1972)

Lewis, F. L., Vrabie, D., Syrmos, V. L.: Optimal control, 3rd edn, John Wiley & Sons (2012)

Von Brasch, T., Byström, J., Lystad, L. P.: Optimal control and the Fibonacci sequence. Discussion papers 674. Statistics Norway (2012)

Sydsæter, K., Hammond, P., Seierstad, A., Strøm, A.: Further mathematics for economic analysis, Financial Times/Prentice Hall (2008)